Run orders and quantitative factors in asymmetrical designs

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Summary

Results on run orders leading to trend-free symmetrical factorial designs are extended to the asymmetrical case, using the character theory of abelian groups. The tools developed equally apply to the construction of designs for quantitative treatment factors with eight or more regularly spaced levels. Abelian group theory can also be used to find minimum-cost run orders for asymmetrical designs, with a cost based on the number of changes of levels between successive runs.

1 Introduction

In factorial designs, runs or underlying experimental units are often partially ordered by their position in time or space. The position's influence is usually controlled by dividing the whole experiment into blocks of consecutive or adjacent units and randomizing out the intra-block position trend. But the quantitative nature of time or location may render blocks an inappropriate or insufficient way of controlling the heterogeneity. It is then advisable, in the analysis, to introduce the position as a covariate and to adjust factorial effects for its linear trend. The square and higher powers of the position can also be introduced to adjust for quadratic or higher degree trends.

In this context, it may be interesting to use a run order that makes important factorial effects orthogonal or nearly orthogonal to the postulated (typically linear or quadratic) polynomial trend. Cheng (1990) and Jacroux (1990) give a good overview of the literature on this kind of designs. In part of it, attention is also paid to the cost of changing factor levels. As outlined by Cheng, "in industrial experiments, it can be difficult, time-consuming and expensive to change factor levels or, after they have been changed, it may take a long time for the system to return to steady state; then the experimenter may want to have as few level changes as possible".

Most of the previous work on run orders of factorial designs only considers the case of symmetric designs in which all the factors have the same number n of levels, where n is a prime power. In this paper, we extend the main results to the asymmetrical case. The main tool is the decomposition of the degrees of freedom of factorial effects associated with the irreducible characters of the group of treatments. This decomposition, first studied by Bailey (1982a), involves contrasts with values in the field $\mathbb{C}$ of complex numbers which are conjugated in pairs. As explained in Kobilinsky (1985a,1990a), Kobilinsky & Monod (1991), their use generally leads to drastic simplifications in calculations.

Using characters and group morphisms also gives an interesting insight into symmetrical designs. This point is illustrated by an example, already considered by Jacroux (1990), of a complete 2^5 design in 2^2 blocks of size 2^3, where the trend may vary from one block to another. It is shown that there is a susbstantial number of designs making main effects orthogonal to blocks and to intra-block trends, and hence a substantial possible amount of randomization to avoid getting too systematic a design.

The technique employed to get trend free designs also gives a very valuable tool to take account of the quantitative nature of some factors other than time and location, in fractional block designs. In section 5, we illustrate this practically important application of the theory. For quantitative factors with 8, 9 levels or more, it provides an alternative, allowing equidistant levels, to the factorial quantitative designs previously considered by Bailey (1982b,1990), Kobilinsky (1985b) and Kobilinsky & Monod (1991). These last ones could not give equidistant levels.

2 Basis of contrasts associated to characters

2.1 The multiplicative group of characters

We suppose there are s treatment factors $A_1, \ldots, A_s$ having $n_1, \ldots, n_s$ levels respectively. The levels of the i^{th} factor are labelled by the elements $0, 1, \ldots, n_i - 1$ of the additive group of integers modulo n_i, denoted by (n_i). The set of $n = n_1 \times \cdots \times n_s$ treatments is then represented by the product group $T = (n_1) \times \cdots \times (n_s)$. The elements of T are column vectors $\mathbf{t} = (t_1, \ldots, t_s)^{\mathrm{t}}$ of dimension s and addition on T is defined componentwise:

$$\begin{pmatrix} t_1 \\ \vdots \\ t_s \end{pmatrix} + \begin{pmatrix} t'_1 \\ \vdots \\ t'_s \end{pmatrix} = \begin{pmatrix} t_1 + t'_1 \\ \vdots \\ t_s + t'_s \end{pmatrix} .$$

A character [1] χ of T is a group morphism from the additive group T into the multiplicative group $\mathbb{C}^*$ of the field $\mathbb{C}$. The set of characters is an abelian group under the product $\chi\xi$ defined by :

$$(\chi\xi)(\mathbf{t}) = \chi(\mathbf{t})\xi(\mathbf{t}) \quad \text{for } \mathbf{t} \in T \tag{1}$$

This multiplicative group is called the *dual group* of T and it is denoted by T^*.

With each factor A_i is associated a particular character, also denoted A_i, defined by:

$$A_i(\mathbf{t}) = \exp\left(\frac{2\pi \mathrm{i}}{n_i} t_i\right) . \tag{2}$$

If each of the n_i levels $0, \ldots, n_i - 1$ of factor A_i is coded by the corresponding power $\rho^0, \ldots, \rho^{n_i-1}$ of the n_i^{th} primitive root of unity $\rho = \exp(2\pi\mathrm{i}/n_i)$, then $A_i(\mathbf{t})$ is simply the coded level of factor A_i for treatment $\mathbf{t}$. There is thus no need to distinguish between the factor A_i and the corresponding character, provided levels are taken as indicated among roots of unity in $\mathbb{C}$.

We shall call $A_1, \ldots, A_s$ the *basic characters*. The following proposition shows that they generate the dual group T^*.

[1]the exact terminology is *irreducible character*. The qualifier irreducible will, however, be dropped since all characters used in this text are irreducible.

Proposition 2.1 *Each character χ can be expressed as a product of powers of the basic characters:*

$$\chi = A_1^{t_1^*} \cdots A_s^{t_s^*} \tag{3}$$

The exponents t_1^*, ... , t_s^* are uniquely defined as elements of (n_1), ... , (n_s) respectively.

Corollary 2.1 *For each element $\mathbf{t}^* = (t_1^*, \dots, t_s^*)^t$ of $(n_1) \times \cdots \times (n_s)$, let :*

$$\eta^{\mathbf{t}^*} = A_1^{t_1^*} \cdots A_s^{t_s^*} \tag{4}$$

Then the mapping $\mathbf{t}^* \mapsto \eta^{\mathbf{t}^*}$ is a group isomorphism from $(n_1) \times \cdots \times (n_s)$ onto T^*. In other words, T^* is the direct sum of the cyclic groups of order n_1, ... , n_s generated by A_1, ... , A_s.

Proof. Let $\mathbf{a}_1$, ... , $\mathbf{a}_s$ be the canonical generators of T:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad \mathbf{a}_s = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} .$$

Then for each treatment $\mathbf{t} = (t_1, \dots, t_s)^{\mathrm{t}} = t_1\mathbf{a}_1 + \cdots + t_s\mathbf{a}_s$, its image under the morphism χ is

$$\chi(\mathbf{t}) = \chi(\mathbf{a}_1)^{t_1} \cdots \chi(\mathbf{a}_s)^{t_s}.$$

Since for each i, $n_i\mathbf{a}_i = 0$, we have $\chi(\mathbf{a}_i)^{n_i} = 1$ and consequently $\chi(\mathbf{a}_i) = \exp\left(\frac{2\pi \mathrm{i}}{n_i} t_i^*\right)$, where the integer t_i^*, defined modulo n_i, can be considered as an element of (n_i). Thus

$$\chi(\mathbf{a}_i)^{t_i} = \exp\left(\frac{2\pi \mathrm{i}}{n_i} t_i t_i^*\right) = \exp\left(\frac{2\pi \mathrm{i}}{n_i} t_i\right)^{t_i^*} = A_i(\mathbf{t})^{t_i^*},$$

and χ is therefore of the form (3).

To prove the unicity of $t_1^*, \ldots, t_s^*$, note that if χ satisfies (3), then $\chi(\mathbf{a}_i) = \exp\left(\frac{2\pi \mathrm{i}}{n_i} t_i^*\right)$, so that t_i^* is uniquely defined modulo n_i.□

Each character χ can thus be considered as a new *factor* or *pseudofactor* derived from the basic ones $A_1, \ldots, A_s$, and we shall sometimes refer to it as the factor or pseudofactor χ.

The characters of T form an orthogonal basis of $\mathbb{C}^T$. More precisely, if χ and ξ are distinct characters, their *inner product*, which is the usual positive definite hermitian form of $\mathbb{C}^T$, is zero:

$$\langle \chi, \xi \rangle = \sum_{\mathbf{t} \in T} \chi(\mathbf{t}) \overline{\xi}(\mathbf{t}) = 0 \qquad \text{if } \chi \neq \xi \, , \tag{5}$$

and

$$\langle \chi, \chi \rangle = \|\chi\|^2 = \sum_{\mathbf{t}} \chi(\mathbf{t}) \overline{\chi}(\mathbf{t}) = n \, . \tag{6}$$

2.2 Factorial effects associated with the characters

With each character χ is associated a linear form $\mathrm{eff}(\chi)$ of the vector $\boldsymbol{\tau}$ of treatment effects, defined by:

$$\mathrm{eff}(\chi) = \langle \boldsymbol{\tau}, \chi \rangle / n \, . \tag{7}$$

This linear form is called the *canonical parameter* or the *factorial effect* associated with χ. If the context makes things clear, no distinction will be made between a character and its associated linear form and $\mathrm{eff}(\chi)$ will be referred to as the parameter or effect χ.

We let H be the $n \times n$ matrix whose columns are the characters of T. If H^* denotes its conjugate transpose, $H^* = \overline{H}^{\mathrm{t}}$, then (5), (6), (7) give :

$$H^* H = n \mathbf{I}_n \, , \tag{8}$$

$$\boldsymbol{\alpha} = H^* \boldsymbol{\tau} / n \, , \tag{9}$$

where $\boldsymbol{\alpha}$ is the vector of canonical parameters : $\boldsymbol{\alpha} = \big(\mathrm{eff}(\chi)\big)_{\chi \in T^*}$. Premultiplying (9) by H, we get

$$\boldsymbol{\tau} = H\boldsymbol{\alpha}\,, \tag{10}$$

which gives as the $\mathbf{t}^{th}$ coordinate of $\boldsymbol{\tau}$

$$\boldsymbol{\tau}(\mathbf{t}) = \sum_{\chi} \chi(\mathbf{t})\,\mathrm{eff}(\chi)\,. \tag{11}$$

It is known and easy to prove that the factorial effects associated with the powers $A_i, \ldots, A_i^{n_i-1}$ span the $n_i - 1$ degrees of freedom of the main effect of factor A_i. Similarly the canonical parameters associated with the $(n_{i_1}-1)\cdots(n_{i_k}-1)$ characters $A_{i_1}^{t^*_{i_1}} \cdots A_{i_k}^{t^*_{i_k}}$, where $t^*_{i_1}, \ldots, t^*_{i_k}$ are all different from 0, span the interaction between the k factors $A_{i_1}, \ldots, A_{i_k}$.

The sum in (11) can be restricted to the subset S^* of characters $\chi \in T^*$ pertaining to non null main effects and interactions. Since for $\chi \neq \overline{\chi}$, the distinct conjugated canonical parameters $\mathrm{eff}(\chi)$ and $\mathrm{eff}(\overline{\chi})$ both belong to the same factorial effect, they are both either in or out model (11). Each such pair in the model can be recombined to get a model with real parameters, as done in Kobilinsky (1985a). However, as pointed out in Kobilinsky (1990a), it is better to work directly with the complex canonical parameters, obtaining thus a linear model

$$E(\mathbf{y}) = X\boldsymbol{\theta}\,, \tag{12}$$

where the vector $\boldsymbol{\theta} = \big(\mathrm{eff}(\chi)\big)_{\chi \in S^*}$ contains pairs of conjugate parameters in $\mathbb{C}$ and X, correspondingly, contains pairs of conjugate columns. The normal equations look like

$$X^* X \hat{\boldsymbol{\theta}} = X^* \mathbf{y}\,, \tag{13}$$

with the conjugate transpose X^* instead of X^{t}. Every classical result from the linear model also holds with this *complex* model. For designs whose construction involves abelian groups, this canonical complex parametrization often leads to a block diagonal matrix X^*X which is much easier to manipulate than any normal equation matrix linked to a real parametrization.

2.3 The group morphism method

If U is a set of units and $\phi : U \to T$ a mapping defining the treatment $\mathbf{t} = \phi(\mathbf{u})$ applied on unit $\mathbf{u}$, it follows from (11) that the column of X associated to a non null canonical parameter $\mathrm{eff}(\chi)$ is the composite map $\chi \circ \phi$. Suppose moreover that U is identified with the group $(m_1) \times \cdots \times (m_r)$ and that

$$\phi(\mathbf{u}) = \Phi \mathbf{u} + \mathbf{t}_0 , \tag{14}$$

where Φ is a group morphism from U into T, and $\mathbf{t}_0$ is a given treatment in T. Such a design is said to be constructed by *the group morphism method.* The treatment effect on unit $\mathbf{u}$ is

$$\boldsymbol{\tau}(\phi(\mathbf{u})) = \sum_{\chi} \chi(\phi(\mathbf{u}))\, \mathrm{eff}(\chi) = \sum_{\chi} \chi \circ \Phi(\mathbf{u})\ \chi(\mathbf{t}_0)\ \mathrm{eff}(\chi) . \tag{15}$$

Since $\chi \circ \Phi$ is clearly a character of U, we have :

$$\boldsymbol{\tau}(\phi(\mathbf{u})) = \sum_{\xi} \xi(\mathbf{u}) \sum_{\substack{\chi \\ \chi \circ \Phi = \xi}} \chi(\mathbf{t}_0)\, \mathrm{eff}(\chi) . \tag{16}$$

The first sum is over the characters ξ of U, the second over those characters χ such that $\chi \circ \Phi = \xi$. The vector $\boldsymbol{\tau} \circ \phi$ of treatment effects on U is therefore

$$\boldsymbol{\tau} \circ \phi = \sum_{\xi} \xi\, \theta(\xi) , \tag{17}$$

where

$$\theta(\xi) = \sum_{\substack{\chi \\ \chi \circ \Phi = \xi}} \chi(\mathbf{t}_0)\ \mathrm{eff}(\chi) . \tag{18}$$

The parameters $\mathrm{eff}(\chi)$ appearing in the sum $\theta(\xi)$ cannot be estimated separately and are said to be *confounded together and with ξ on U.* The associated characters are the elements of the inverse image of ξ by the mapping $\chi \to \chi \circ \Phi$ from T^* into U^*. This mapping is a morphism which is called the *dual of* Φ and denoted by Φ^*. If we identify the characters with the corresponding parameters, this result can be reexpressed as follows:

Proposition 2.2 *The parameters confounded with $\xi \in U^*$ are the elements of the reciprocal image ${\Phi^*}^{-1}(\xi)$.*

2.4 Representation of the dual by an additive group

Let M be a common multiple of $n_1, \ldots, n_s$. Then $M\mathbf{t} = 0$, hence $\chi(\mathbf{t})^M = \chi(M\mathbf{t}) = 1$ for every $\mathbf{t} \in T$ and $\chi \in T^*$. The image of any character is therefore in the multiplicative group of M^{th} roots of unity in $\mathbb{C}^*$. This group being isomorphic to the cyclic additive group (M), we can as well define the dual T^* as the additive group $\mathrm{Mor}\Big(T,(M)\Big)$ of morphisms from T into (M). If $\zeta \in \mathrm{Mor}\Big(T,(M)\Big)$ and $\eta = \exp\left(\frac{2\pi \mathrm{i}}{M}\right)$, the character χ corresponding to ζ is defined by

$$\chi(\mathbf{t}) = \eta^{\zeta(\mathbf{t})} = \exp\left(\frac{2\pi \mathrm{i}}{M}\zeta(\mathbf{t})\right) \ . \tag{19}$$

Both the multiplicative and additive notations have their own advantages and are used. In order to avoid a multiplicity of symbols, the same symbols $A_1, \ldots, A_s$ are employed to denote either the basic characters, or the corresponding morphisms from T into (M), defined instead of (2) by

$$A_i(\mathbf{t}) = \frac{M}{n_i} t_i. \tag{20}$$

This is not a source of confusion since the context always makes clear which of the two notations, the multiplicative or the additive one, is used.

In additive notation, proposition 2.1 says that any element ζ of the dual T^* can be represented uniquely as a linear combination

$$\zeta = t_1^* A_1 + \cdots + t_s^* A_s, \tag{21}$$

where $t_1^* \in (n_1), \ldots, t_s^* \in (n_s)$. If $\mathbf{t}^* = (t_1^*, \ldots, t_s^*)^{\mathrm{t}}$, then

$$\zeta(\mathbf{t}) = [\mathbf{t}^*, \mathbf{t}], \tag{22}$$

where $[\mathbf{t}^*, \mathbf{t}]$ is the (M)-bilinear form from $T^* \times T$ into (M) given by

$$[\mathbf{t}^*, \mathbf{t}] = \sum_i \frac{M}{n_i} t_i^* t_i \ . \tag{23}$$

The dual T^* can be identified with the product group $(n_1)\times\cdots\times(n_s)$ by means of the isomorphism $\mathbf{t}^* \mapsto \zeta$. In practice, the element $\mathbf{t}^*$ is completely identified with the corresponding morphism $\zeta = t_1^* A_1 + \cdots + t_s^* A_s$. We can thus say that the canonical generators of T^* are the factors $A_1, \ldots, A_s$ (which take their values in (M)), write an equality like $\mathbf{t}^* = t_1^* A_1 + \cdots + t_s^* A_s$ and speak of the morphism $\mathbf{t}^*$, whose value on $\mathbf{t}$ is $[\mathbf{t}^*, \mathbf{t}]$. We shall also use the terminology *factor* or *pseudofactor* to refer to these morphisms into (M).

The following identities connecting $\mathbf{t}^*$ and the corresponding character $\eta^{\mathbf{t}^*}$ can be useful:

$$\eta^{\mathbf{t}^*}(\mathbf{t}) = \eta^{[\mathbf{t}^*,\mathbf{t}]} = \exp\left(\frac{2\pi \mathrm{i}}{M}[\mathbf{t}^*, \mathbf{t}]\right). \tag{24}$$

2.5 Matrix representation of morphisms

Let Φ be a morphism from $U = (m_1) \times \cdots \times (m_r)$ into T, and let $\mathbf{u}_1, \ldots, \mathbf{u}_r$ be the canonical generators of U. For each unit $\mathbf{u} = (u_1, \ldots, u_r)^{\mathrm{t}} = u_1\mathbf{u}_1 + \cdots + u_r\mathbf{u}_r$, its image under the morphism Φ is $u_1\Phi(\mathbf{u}_1) + \cdots + u_r\Phi(\mathbf{u}_r)$. Since $m_i\mathbf{u}_i = 0$ for each i, the treatments $\mathbf{t}_i = \Phi(\mathbf{u}_i)$ satisfy $m_i\mathbf{t}_i = 0$. Conversely, if $\mathbf{t}_1, \ldots, \mathbf{t}_r$ are treatments in T satisfying

$$m_i\mathbf{t}_i = 0 \quad \text{for } i = 1, \ldots, r, \tag{25}$$

then the map Φ defined by

$$\Phi(\mathbf{u}) = u_1\mathbf{t}_1 + \cdots + u_r\mathbf{t}_r \tag{26}$$

is a morphism from U to T. This morphism Φ can be represented by the matrix having $\mathbf{t}_1, \ldots, \mathbf{t}_r$ as columns, which is denoted by Φ too. The image of $\mathbf{u}$ under Φ is then the matrix product $\Phi\mathbf{u}$.

Consider now the dual $\Phi^* : T^* \to U^*$. Choosing for M a common multiple of $n_1, \ldots, n_s$ and *at the same time* of $m_1, \ldots, m_r$, we can identify T^* with $(n_1) \times \cdots \times (n_s)$ and U^* with $(m_1) \times \cdots \times (m_r)$. The image $\Phi^*\mathbf{t}^*$ of an element $\mathbf{t}^* \in T^*$ is then defined by :

$$[\Phi^*\mathbf{t}^*, \mathbf{u}] = [\mathbf{t}^*, \Phi\mathbf{u}], \quad \text{for all } \mathbf{u} \in U \tag{27}$$

The dual Φ^* can be represented by a matrix exactly as Φ. The columns of this matrix $\Phi^* A_1 = A_1 \circ \Phi, \ldots, \Phi^* A_s = A_s \circ \Phi$ are in $U^* = (m_1) \times \cdots \times (m_r)$.

More details on the properties of Φ and Φ^* can be found in Kobilinsky (1985a).

3 Trend free design

Let $\phi : U \to T$ be a map assigning a treatment $\mathbf{t} = \phi(\mathbf{u})$ to each unit $\mathbf{u}$ in U, and $f : U \to \mathbb{R}$ a quantitative map specifying for instance an order on U. The expectation of the response y on unit $\mathbf{u}$ is assumed to be:

$$E(y) = \sum_{j=0}^{k} \beta_j f(\mathbf{u})^j + \sum_{\chi} \chi\big(\phi(\mathbf{u})\big) \ \operatorname{eff}(\chi) \tag{28}$$

The first sum describes a trend of order k. The second one, which is over the characters χ such that $\operatorname{eff}(\chi) \neq 0$, gives the effect of treatment $\mathbf{t} = \phi(\mathbf{u})$. The columns of the associated X matrix in $\mathbb{R}^U$ and $\mathbb{C}^U$ are the powers f^j and the composite maps $\chi \circ \phi$. It is interesting to know conditions ensuring the orthogonality between these two kinds of columns, that is conditions under which the inner products $\langle \chi \circ \phi, f^j \rangle$ are zero. Indeed, if the treatment effects of interest are orthogonal to the trend, they are estimated without any loss of precision due to the adjustment for the trend. This motivates the following definition :

Definition 3.1 (k-trend orthogonality) *A map $\xi : U \to \mathbb{C}$ is said to be* k-trend orthogonal *(or* k-trend free*) with respect to $f : U \to \mathbb{R}$ if $\langle \xi, f^j \rangle = 0$ for $0 \leq j \leq k$.*

Since $f^0 = \mathbf{1}$, the 0-trend orthogonality of ξ means that $\langle \xi, \mathbf{1} \rangle = 0$, i.e. that ξ is centered (we assume $x^0 = 1$ even if $x = 0$). No special property is required for ξ to be k-trend orthogonal if $k < 0$, since it is impossible to have $0 \leq j \leq k$ in that case. It will nevertheless be convenient to consider this case too in recurrent processes.

The following proposition gives a condition of trend orthogonality for characters.

Proposition 3.1 *Let $U = (m_1) \times \cdots \times (m_r)$ and f be a* trend *function on U defined by $f(\mathbf{u}) = u_1 K_1 + \cdots + u_r K_r$, where $K_1, \ldots, K_r$ are real constants* *and, for each* $j = 1, \dots, r$*, the integer* u_j *is the representative between* 0 *and* $m_j - 1$ *of the* j^{th} *coordinate of* $\mathbf{u}$*. Let* $U_1, \dots, U_r$ *be the basic characters in* U^**. Then the decomposition of* f *on the orthogonal basis of characters of* U *gives:*

$$f = \lambda \mathbf{1} + \sum_{j=1}^{r} K_j \sum_{u_j^*=1}^{m_j-1} \frac{1}{\exp\left(\frac{2\pi \mathrm{i} u_j^*}{m_j}\right) - 1} U_j^{-u_j^*} \tag{29}$$

with $\lambda = \langle f, \mathbf{1} \rangle / \langle \mathbf{1}, \mathbf{1} \rangle$*.*

Note that any *natural* order can be described by a trend function of the form assumed. For instance, if the elements $\mathbf{u} = (u_1, \dots, u_r)$ of U are sorted according to the value of u_r, then for a constant u_r according to the value of u_{r-1} and so on, we have :

$$f(\mathbf{u}) = u_1 + m_1 u_2 + m_2 m_1 u_3 + \cdots + m_{r-1} \cdots m_1 u_r$$

Corollary 3.1 *With the notations of proposition 3.1, if* $\mathbf{u}^* = (u_1^*, \dots, u_r^*)^t$ *in* U^* *has* k *non-zero coordinates* u_j^**, the associated character* $\xi = \eta^{\mathbf{u}^*}$ *is* $(k-1)$*-trend orthogonal with respect to* f *and*

$$\langle \xi, f^k \rangle = N k! \prod_{\substack{j \\ \rho_j \neq 1}} \frac{K_j}{\rho_j - 1}$$

where $N = m_1 \times \cdots \times m_r$ *is the number of units, and* $\rho_j = \exp(2\pi \mathrm{i} u_j^* / m_j)$*.*

The proof of proposition 3.1 and corollary 3.1 uses the following lemma:

Lemma 3.1 *If* $\rho^m = 1$ *and* $\rho \neq 1$*, then*

$$(i) \quad \sum_{u=0}^{m-1} \rho^u = 0 \qquad\qquad (ii) \quad \sum_{u=0}^{m-1} u\, \rho^u = \frac{m}{\rho - 1}$$

Proof:

(i) $\displaystyle\sum_{u=0}^{m-1} \rho^u = \frac{\rho^m - 1}{\rho - 1} = 0$

(ii) For $x \in \mathbb{C}$ and different from 0 and 1, we have:

$$\begin{aligned}\sum_{u=0}^{m-1} u\, x^u = x \sum_{u=0}^{m-1} u\, x^{u-1} &= x\frac{d}{dx}\left(\sum_{u=0}^{m-1} x^u\right) = x\frac{d}{dx}\left(\frac{x^m-1}{x-1}\right) \\ &= x\ \frac{mx^{m-1}(x-1)-(x^m-1)}{(x-1)^2}\end{aligned}$$

Replacing x by ρ in the last expression and using the equality $\rho^m = 1$, we get the result of the lemma. □

Proof of proposition 3.1.

The following proof is the straightforward extension of that given by Cheng, Jacroux (1988) for two-level factors.

Let $\xi = \eta^{\mathbf{u}^*}$ and

$$\rho_j = \exp\left(\frac{2\pi \mathrm{i} u_j^*}{m_j}\right) \quad \text{for } j = 1, \dots, r.$$

Then $\xi(\mathbf{u}) = \rho_1^{u_1} \cdots \rho_r^{u_r}$, hence

$$\langle \xi, f\rangle = \sum_{u_1,\dots,u_r} \rho_1^{u_1}\cdots\rho_r^{u_r} \sum_{i=1}^{r} u_i K_i = \sum_{i=1}^{r}\left(\sum_{u_i=0}^{m_i-1} u_i K_i \rho_i^{u_i}\right)\prod_{j\neq i}\left(\sum_{u_j=0}^{m_j-1}\rho_j^{u_j}\right)$$

To each of the k non-zero coordinates u_j^* of $\mathbf{u}^*$ is associated a m_j-root of unity ρ_j which is different from 1. Hence if $k \geq 2$, for each term in the sum over i, at least one ρ_j with $j \neq i$ is different from 1 and therefore $\langle \xi, f\rangle = 0$.

If $k = 1$ and u_l^* is the only non zero coordinate of $\mathbf{u}^*$, then all terms in the sum over i are zero but the l^{th} one, which is

$$\frac{m_l K_l}{\rho_l - 1}\prod_{j\neq l} m_j = \frac{NK_l}{\rho_l - 1}$$

Thus

$$\langle \xi, f\rangle = \left\langle U_l^{u_l^*}, f\right\rangle = \frac{NK_l}{\rho_l - 1} = \frac{NK_l}{\exp\left(\frac{2\pi \mathrm{i} u_l^*}{m_l}\right) - 1}.$$

The decomposition of f on the orthogonal basis of characters of U gives $f = \sum_\xi \langle \xi, f \rangle \overline{\xi}/N$. The result of the proposition is obtained by replacing the inner products $\langle \xi, f \rangle$ by their values. □

Proof of corollary 3.1.
Using the bilinearity of the coordinatewise product in $\mathbb{C}^U$ to develop the j^{th} power of decomposition (29), we get the expression of f^j as a linear combination of the characters of U. This decomposition cannot involve products $\overline{\xi} = U_{j_1}^{-u^*_{j_1}} \cdots U_{j_k}^{-u^*_{j_k}}$ with k terms if $k > j$. Such products are therefore $(k-1)$-trend orthogonal. If $j = k$, the coefficient of such a product is

$$k! \frac{K_{j_1}}{\rho_{j_1} - 1} \cdots \frac{K_{j_k}}{\rho_{j_k} - 1} .$$

This proves the corollary. □

Corollary 3.2 *Let Φ be a group morphism from $U = (m_1) \times \cdots \times (m_r)$ into T. We suppose, as in the proposition, that the trend function f is defined by $f(\mathbf{u}) = u_1 K_1 + \cdots + u_r K_r$ and let $\phi(\mathbf{u}) = \Phi\mathbf{u} + \mathbf{t}_0$. Then if $\chi = \eta^{\mathbf{t}^*}$ is a character of T, the induced map $\chi \circ \phi = \chi(\mathbf{t}_0)\eta^{\Phi^*\mathbf{t}^*}$ on U is $(k-1)$-trend orthogonal if and only if $\Phi^*\mathbf{t}^*$ has at least k non-zero coordinates.*

Cheng (1985) and Coster and Cheng (1988) gave another proof of Proposition 3.1 for symmetrical designs, which uses an iterative process of construction of ϕ and f. Though the examples in their paper only deal with the case of regular fractions, that is fractions defined by a ϕ satisfying (14), the iterative process mentioned can start with a more general design as exemplified by John (1990) for seven 3-level factors in 18 runs. So, in order to keep the same degree of generality, we now generalize this process to asymmetrical designs.

We start with maps $\phi : U \to T$ and $f : U \to \mathbb{R}$ which are not necessarily like those of Corollary 3.2. They are extended to the cartesian product $\tilde{U} = U \times (m)$ by the maps $\tilde{\phi}$ and $\tilde{f}$ defined, for $\mathbf{u} \in U$ and $v \in (m)$, by

$$\tilde{\phi}(\mathbf{u}, v) = \phi(\mathbf{u}) + v\mathbf{t} \tag{30}$$

$$\tilde{f}(\mathbf{u}, v) = f(\mathbf{u}) + v\ K \tag{31}$$

where K is a real constant, $0 \le v \le m-1$, and $\mathbf{t}$ is a given treatment in T satisfying

$$m\mathbf{t} = 0 \tag{32}$$

Let χ be a character of T. The next proposition shows that if $\chi \circ \phi$ is $(k-1)$-trend orthogonal, its extension $\chi \circ \tilde{\phi}$ to $\tilde{U}$ is equally $(k-1)$-trend orthogonal and that it is moreover k-trend orthogonal if $\chi(\mathbf{t}) \neq 1$. Besides, it gives a way to compute $\left\langle \chi \circ \tilde{\phi}, \tilde{f}^k \right\rangle$, if $\chi(\mathbf{t}) = 1$, or $\left\langle \chi \circ \tilde{\phi}, \tilde{f}^{k+1} \right\rangle$, if $\chi(\mathbf{t}) \neq 1$. Note that the same notation $\langle\ \rangle$ will be used here for the inner products on $\mathbb{C}^U$ and $\mathbb{C}^{\tilde{U}}$.

We let $\xi = \chi \circ \phi$ and $\tilde{\xi} = \chi \circ \tilde{\phi}$. Indeed, the proposition only relies on the following relation between ξ and $\tilde{\xi}$

$$\tilde{\xi}(\mathbf{u}, v) = \xi(\mathbf{u})\rho^v \tag{33}$$

where ρ is a m^{th} root of unity:

$$\rho^m = 1\ . \tag{34}$$

Relation (33) is an immediate consequence of (30) and (32), if we take $\rho = \chi(\mathbf{t})$.

Proposition 3.2 *Assume* $\langle \xi, f^j \rangle = 0$ *for* $0 \le j \le k-1$ *where* $k \ge 0$. *If* $\tilde{f}$ *and* $\tilde{\xi}$ *are extensions of* f *and* ξ *to* $\tilde{U}$ *satisfying (31) and (33) respectively, then:*

$$\begin{array}{lcll}
\left\langle \tilde{\xi}, \tilde{f}^j \right\rangle & = & 0 & \textit{for} \quad 0 \le j \le k-1 \\
\left\langle \tilde{\xi}, \tilde{f}^k \right\rangle & = & 0 & \textit{if} \quad \rho \neq 1 \\
\left\langle \tilde{\xi}, \tilde{f}^k \right\rangle & = & m \left\langle \xi, f^k \right\rangle & \textit{if} \quad \rho = 1 \\
\left\langle \tilde{\xi}, \tilde{f}^{k+1} \right\rangle & = & \dfrac{(k+1)mK}{\rho - 1} \left\langle \xi, f^k \right\rangle & \textit{if} \quad \rho \neq 1
\end{array}$$

Proof: With the hypothesis of the proposition

$$\begin{aligned}
\left\langle \tilde{\xi}, \tilde{f}^j \right\rangle &= \sum_{\mathbf{u}\in U} \sum_{v=0}^{m-1} \xi(\mathbf{u})\rho^v \ (f(\mathbf{u}) + vK)^j \\
&= \sum_{\mathbf{u}\in U} \sum_{v=0}^{m-1} \xi(\mathbf{u})\rho^v \ \sum_{i=0}^{j} \binom{j}{i} \ f(\mathbf{u})^i (vK)^{j-i} \\
&= \sum_{v=0}^{m-1} \rho^v \sum_{i=0}^{j} \binom{j}{i} \ (vK)^{j-i} \sum_{\mathbf{u}\in U} \xi(\mathbf{u}) f(\mathbf{u})^i \\
&= \sum_{v=0}^{m-1} \rho^v \sum_{i=0}^{j} \binom{j}{i} \ (vK)^{j-i} \left\langle \xi, f^i \right\rangle
\end{aligned}$$

- if $0 \le j \le k-1$, then all inner products on the right are zero and so is $\left\langle \tilde{\xi}, \tilde{f}^j \right\rangle$.
- if $j = k$, the only non-zero term in the right sum, obtained when $i = k$, is $\left\langle \xi, f^k \right\rangle$. Thus:

$$\left\langle \tilde{\xi}, \tilde{f}^k \right\rangle = \left(\sum_{v=0}^{m-1} \rho^v \right) \left\langle \xi, f^k \right\rangle$$

  * if $\rho \neq 1$ then $\sum_{v=0}^{m-1} \rho^v = 0$, hence $\left\langle \tilde{\xi}, \tilde{f}^k \right\rangle = 0$.
  * if $\rho = 1$ then $\sum_{v=0}^{m-1} \rho^v = m$, hence $\left\langle \tilde{\xi}, \tilde{f}^k \right\rangle = m \ \left\langle \xi, f^k \right\rangle$.

- for $j = k+1$ and $\rho \neq 1$, we find

$$\begin{aligned}
\left\langle \tilde{\xi}, \tilde{f}^{k+1} \right\rangle &= \sum_{v=0}^{m-1} \rho^v \left[\binom{k+1}{k} \ vK \left\langle \xi, f^k \right\rangle + \left\langle \xi, f^{k+1} \right\rangle \right] \\
&= (k+1)K \left\langle \xi, f^k \right\rangle \sum_{v=0}^{m-1} v \ \rho^v + \left\langle \xi, f^{k+1} \right\rangle \sum_{v=0}^{m-1} \rho^v \\
&= \frac{(k+1)mK}{\rho - 1} \left\langle \xi, f^k \right\rangle \ . \quad \square
\end{aligned}$$

Example 1 . Suppose that the experimental units of a complete $2^3 \times 3^2$ design are totally ordered by time or space. We identify them with the elements of the group $U = (2)^3 \times (3)^2$, in such a way that their order position is given by a trend function f of the type described by Proposition 3.1, for instance:

$$f(\mathbf{u}) = u_1 + 2\,u_4 + 2 \times 3\,u_5 + 2 \times 3 \times 3\,u_2 + 2 \times 3 \times 3 \times 2\,u_3 \qquad (35)$$

The assignment of treatments in $T = (2)^3 \times (3)^2$ to units in U is defined by the morphism $\Phi : U \to T$ whose dual matrix is

$$\Phi^* = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \qquad (36)$$

It then follows from Corollary 3.2 that all main effects and two factor interactions between one 3-level factor and one 2-level factor are linear trend free. Recombining each 3-level factor with a 2-level one, we get a complete $6 \times 6 \times 2$ design whose main effects are linear trend orthogonal.

The image $\tilde{A}_3 = (0, 1, 1, 0, 0)^{\mathrm{t}}$ of the third factor A_3 by Φ^* gives the level $3(u_2 + u_3)$ in the cyclic group (6) to unit $\mathbf{u} = (u_1, \ldots, u_5)^{\mathrm{t}}$. Hence, with the order defined by (35), the factor A_3 remains constant on series of $2 \times 3 \times 3$ consecutive units. Therefore, if the trend does not remain linear on the whole sequence of 72 units, the main effect of A_3 can be seriously biased.

4 Trend free block designs

On long sequence of experimental units over time or space, it is difficult to contend that the trend, if it exists, can be adequately described by a linear or quadratic function of the unit position. As shown by example 1, it can then be dangerous to restrict randomization in order to get important effects linear or quadratic trend orthogonal. It is therefore useful to know how to adapt the technique to block designs. Indeed, if the size of blocks is moderate, it can be reasonable to assume that the trend is linear or quadratic within each block.

Two different models may be assumed for block designs. In the first one, the trend is block dependent, while in the second one the trend is the same within each block:

$$\text{block-dependent trend :} \qquad E(y) = \sum_{j=0}^{k} \beta_{j\mathbf{b}} f(\mathbf{u})^j + \sum_{\chi} \chi\big(\phi(\mathbf{u})\big) \ \text{eff}(\chi) \tag{37}$$

$$\text{block-independent trend :} \quad E(y) = \alpha_{\mathbf{b}} + \sum_{j=0}^{k} \beta_j f(\mathbf{u})^j + \sum_{\chi} \chi\big(\phi(\mathbf{u})\big) \ \text{eff}(\chi) \tag{38}$$

The function f here gives the position within the block $\mathbf{b}$ containing unit $\mathbf{u}$.

The block-independent trend (38) will be used only if the block and position factors are crossed. For instance, in an experiment involving a cooking, $f(\mathbf{u})$ could be the position of the unit in the oven, while a block is the set of units cooked at the same time. On the other hand, with the block-dependent trend, the position is considered as nested within blocks.

To compute inner products between the columns of the corresponding X matrix, we assume that:

- $\phi(\mathbf{u}) = \Phi\mathbf{u} + \mathbf{t}_0$ where Φ is a morphism from U into T as in Corollary 3.2.
- The blocks are the cosets of a subgroup W of U, which is called the *principal block*.

We now consider separately the two models.

4.1 Block dependent trend

The positions of the units are defined, *independently for each block*, in the following manner.

The principal block W is decomposed as a direct sum of cyclic groups and we let Δ be a matrix whose columns are generators of these cyclic groups. If $l_1, \ldots, l_q$ are their orders, Δ is thus a $r \times q$ matrix defining an injective morphism from the product group $V = (l_1) \times \cdots \times (l_q)$ into $U = (m_1) \times \cdots \times (m_r)$, whose image is W.

A representative $\mathbf{w} \in U$ of the block $\mathbf{b}$ under consideration is then selected and the elements of the blocks are identified with those of V by the injective mapping $\delta : \mathbf{v} \mapsto \Delta\mathbf{v} + \mathbf{w}$ from V into U.

The natural order of the product group V then induces an order on the elements of block $\mathbf{b}$. More precisely, the rank $g(\mathbf{v})$, for the natural order, of the element $\mathbf{v} = (v_1, \ldots, v_q)$ in V is

$$g(\mathbf{v}) = v_1 K_1 + \cdots + v_q K_q \quad \text{where } K_i = \prod_{j<i} l_j, \tag{39}$$

and the induced position $f(\mathbf{u})$ of the element $\mathbf{u} = \delta(\mathbf{v})$ of block $\mathbf{b}$ is

$$f(\mathbf{u}) = f(\delta(\mathbf{v})) = g(\mathbf{v}) \tag{40}$$

The column associated with the parameter $\beta_{j\mathbf{b}}$ in (37) has 0 for each unit outside block $\mathbf{b}$ and $f(\mathbf{u})^j = g(\mathbf{v})^j$ for unit $\mathbf{u} = \delta(\mathbf{v})$ of block $\mathbf{b}$. The coefficient of eff(χ) for this unit is :

$$\chi\Big(\phi(\delta(\mathbf{v}))\Big) = \chi\Big(\Phi(\Delta\mathbf{v} + \mathbf{w}) + \mathbf{t}_0\Big) = \chi(\mathbf{t}_0)\ \chi \circ \Phi(\mathbf{w})\ \chi \circ \Phi \circ \Delta(\mathbf{v}) \tag{41}$$

Therefore, the inner product between the columns associated to $\beta_{j\mathbf{b}}$ and eff(χ) is:

$$\langle \chi \circ \phi \circ \delta, g^j \rangle = \chi(\mathbf{t}_0)\ \chi \circ \Phi(\mathbf{w})\ \langle \chi \circ \Phi \circ \Delta, g^j \rangle \tag{42}$$

If $\chi = \eta^{\mathbf{t}^*}$, it follows from Corollary 3.1 that, if $\Delta^*\Phi^*\mathbf{t}^*$ has at least k non-zero coordinates, the inner product (42) is zero for $j \leq k-1$. Consequently, factor χ is then $(k-1)$-trend orthogonal within block $\mathbf{b}$.

Remarks on the choice of Δ and $\mathbf{w}$.

If Δ_1 is obtained from Δ by a permutation of columns, the vector $\Delta_1^*\Phi^*\mathbf{t}^*$ is obtained from $\Delta^*\Phi^*\mathbf{t}^*$ by the same permutation on the coordinates. Hence, for any choice of Δ leading to the desired degree of trend-orthogonality of the important effects, a number of other satisfying matrices Δ are obtained simply by permuting the columns.

One may wonder whether the different choices of Δ and $\mathbf{w}$ really give different orders for the units of the block $\mathbf{b}$. The next proposition shows that the answer is yes. Indeed let Δ_1,$\mathbf{w}_1$ and Δ_2,$\mathbf{w}_2$ be two possible choices. Let $V_1 = (k_1) \times \cdots \times (k_p)$ and $V_2 = (l_1) \times \cdots \times (l_q)$ be the corresponding products of cyclic groups and δ_1, δ_2 the corresponding mappings into the block $\mathbf{b}$. Then,

Proposition 4.1 *If the orders induced on the block* $\mathbf{b}$ *by the natural orders of the products* V_1 *and* V_2 *are identical, then* $\mathbf{w}_1 = \mathbf{w}_2$, $\Delta_1 = \Delta_2$ *and therefore* $p = q$, $k_1 = l_1, \ldots, k_q = l_q$.

Proof. By hypothesis, the elements having the same rank for the natural order in V_1 and V_2 must have the same images by δ_1 and δ_2. The first elements are 0 in both groups. Hence $\mathbf{w}_1 = \delta_1(0) = \delta_2(0) = \mathbf{w}_2$. The second elements are

$$\underbrace{(1,0,0,\ldots,0)^t}_{\text{length p}} \quad \underbrace{(1,0,0,\ldots,0)^t}_{\text{length q}} .$$

Hence the first columns of Δ_1 and Δ_2 are equal and $k_1 = l_1$. But then, the $(k_1+1)^{th}$ elements are :

$$\underbrace{(0,1,0,\ldots,0)^t}_{\text{length p}} \quad \underbrace{(0,1,0,\ldots,0)^t}_{\text{length q}} .$$

Hence the second columns of Δ_1 and Δ_2 are equal and $k_2 = l_2$, and so on ...
□

Example 2 . Complete 2^n designs in blocks of size 2^3, with main effects unconfounded and linear trend orthogonal within blocks

We take $\Phi^* = \mathbf{I}_n$. To define W, we select a $n \times 3$ full rank matrix Δ_0 with at least two 1 by row, for instance with $n = 5$

$$\Delta_0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} . \tag{43}$$

The principal block W is then generated by the columns of Δ_0, which in multiplicative notation are *bcde*, *acde*, *abcde*. The other blocks can be deduced by multiplying by c, d, e respectively the elements of the principal block.

Here $\Delta_0^* = \Delta_0^t$. Since no column of Δ_0^* is 0, main effects are unconfounded with the blocks.

To define the order within the principal block, we have to choose a 5×3 matrix Δ which has the same image W as Δ_0 and equally at least two 1 by row. A backtrack search gives, up to a permutation of columns, the three matrices :

$$\Delta_0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \Delta_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \Delta_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{44}$$

Each one leads, by permutation of its columns, to 6 possible matrices Δ, and there are thus $3 \times 6 = 18$ possible matrices Δ. For each choice of Δ, there are 8 possible choices for $\mathbf{w}$, which gives a total of $18 \times 8 = 144$ possible orders for the principal block. The 144 possible orders of each other block are then deduced by adding a representative of the block (multiplying in multiplicative notation). The orders can be chosen independently for the different blocks. Table 1 gives four possible orders for the principal block. Multiplying by c, d, e respectively the last three ones, we get in table 2 one of the 144^4 possible designs.

Note that the properties of the designs deduced from Δ_0, Δ_1, Δ_2 are not exactly similar if one is also concerned by a possible quadratic trend or by some two-factor interactions.

generator				associated order							
1	2	3	**w**								
ab	*abcde*	*bcde*	*abcde*	*abcde*	*cde*	1	*ab*	*a*	*b*	*bcde*	*acde*
abcde	*bcde*	*ab*	*bcde*	*bcde*	*a*	1	*abcde*	*acde*	*b*	*ab*	*cde*
acde	*bcde*	*abcde*	*b*	*b*	*abcde*	*cde*	*a*	*acde*	1	*ab*	*bcde*
abcde	*ab*	*acde*	1	1	*abcde*	*ab*	*cde*	*acde*	*b*	*bcde*	*a*

Table 1: selection of 4 orders for the principal block

It is unfortunately impossible in the asymmetrical case to get a similar design with fewer than 36 units per block. Let again $U = T$, $\Phi = \mathbf{I}_r$ and assume that the numbers of levels m_1, m_2 of the first two factors A and

block	representative	order							
1	1	*abcde*	*cde*	1	*ab*	*a*	*b*	*bcde*	*acde*
2	*c*	*bde*	*ac*	*c*	*abde*	*ade*	*bc*	*abc*	*de*
3	*d*	*bd*	*abce*	*ce*	*ad*	*ace*	*d*	*abd*	*bce*
4	*e*	*e*	*abcd*	*abe*	*cd*	*acd*	*be*	*bcd*	*ae*

Table 2: 2^5 in 4 ordered blocks. Main effects are orthogonal to block effects and linear trend orthogonal within blocks

B are multiples of 2 and 3 respectively. Then we can find elements $\mathbf{t}_1^* = (t_1^*, 0, \ldots, 0)$ and $\mathbf{t}_2^* = (0, t_2^*, 0, \ldots, 0)$ pertaining to the main effects A and B and having period 2 and 3 respectively. If these main effects are to be linear trend orthogonal, $\Delta^*\mathbf{t}_1^*$ and $\Delta^*\mathbf{t}_2^*$ must have at least 2 non-zero coordinates. This implies that, among the orders $l_1, \ldots, l_q$, at least two are multiples of 2 and, similarly, at least two are multiples of 3. Hence the order of the product group $V = (l_1) \times \cdots \times (l_q)$ is a multiple of $2^2 \times 3^2 = 36$.

4.2 Block independent trend

It is now assumed that U can be decomposed into the direct sum of a principal block V and another subgroup B. We choose decompositions $V = (m_1) \times \cdots \times (m_q)$, $B = (m_{q+1}) \times \cdots \times (m_r)$ and select the corresponding decomposition $U = V \times B = (m_1) \times \cdots \times (m_r)$ of U.

On unit $\mathbf{u} = \begin{pmatrix} \mathbf{v} \\ \mathbf{b} \end{pmatrix}$, the block, intra-block position and treatment are defined by :

$$\text{block :} \qquad \Psi(\mathbf{u}) = \Psi \begin{pmatrix} \mathbf{v} \\ \mathbf{b} \end{pmatrix} = \mathbf{b} \ , \tag{45}$$

$$\text{position :} \qquad f(\mathbf{u}) = f \begin{pmatrix} \mathbf{v} \\ \mathbf{b} \end{pmatrix} = g(\mathbf{v}) \ , \tag{46}$$

$$\text{treatment :} \quad \Phi\mathbf{u} + \mathbf{t}_0 = (\Phi_V, \Phi_B) \begin{pmatrix} \mathbf{v} \\ \mathbf{b} \end{pmatrix} + \mathbf{t}_0 = \Phi_V\mathbf{v} + \Phi_B\mathbf{b} + \mathbf{t}_0 \ , \tag{47}$$

where g is defined, as in (39), by

$$g(\mathbf{v}) = v_1 K_1 + \cdots + v_q K_q \quad \text{where } K_i = \prod_{j<i} m_j, \tag{48}$$

and (Φ_V, Φ_B) is the partition of Φ into matrices of sizes $s \times q$ and $s \times (r-q)$ respectively.

The inner product between the columns associated to β_j and $\mathrm{eff}(\chi)$ in model (38) is then

$$\sum_{\mathbf{b}\in B} \sum_{\mathbf{v}\in V} g(\mathbf{v})^j \chi(\Phi_V \mathbf{v} + \Phi_B \mathbf{b} + \mathbf{t}_0) = \chi(\mathbf{t}_0) \langle \chi \circ \Phi_B, \mathbf{1} \rangle \langle \chi \circ \Phi_V, g^j \rangle \tag{49}$$

Consequently, if $\chi = \eta^{\mathbf{t}^*}$, this inner product is zero

- either if $\Phi_B^* \mathbf{t}^* \neq 0$, that is $\mathbf{t}^* \notin \mathrm{Ker}\, \Phi_B^*$,
- or if $\Phi_V^* \mathbf{t}^*$ has at least $j+1$ non zero coordinates.

The subgroup confounded with the blocks in $U^* = V^* \times B^*$ is the kernel of the projection onto V^*, whose matrix is $(\mathbf{I}_q, 0)$. The subgroup of treatment effects confounded with the blocks is therefore the kernel of $(\mathbf{I}_q, 0)\Phi^* = \Phi_V^*$.

Example 3 . $2 \times 2 \times 3 \times 2 \times 3$ **in blocks of size** $2 \times 2 \times 3$

We let $V = (2) \times (2) \times (3)$, $B = (2) \times (3)$, $U = V \times B = (2) \times (2) \times (3) \times (2) \times (3)$, $T = (2) \times (2) \times (3) \times (2) \times (3)$ and

$$\Phi^* = \begin{array}{c} \begin{matrix} (2) & (2) & (3) & (2) & (3) \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix} \end{array} \begin{matrix} (2) \\ (2) \\ (3) \\ (2) \\ (3) \end{matrix} .$$

We then have

$$\Phi_V^* = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{matrix} (2) \\ (2) \\ (3) \end{matrix} \quad \Phi_B^* = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix} \begin{matrix} (2) \\ (3) \end{matrix}$$

Call $A_1, \dots, A_5$ the canonical generators of T^* (and the associated factors). Then it is easy to see that $\operatorname{Ker}\Phi_V^*$ is generated by $A_1 + A_2 + A_4 = (1,1,0,1,0)^{\mathrm{t}}$ and $A_3 + 2A_5 = (0,0,1,0,2)^{\mathrm{t}}$. Thus, apart from the two degrees of freedom $A_3 + 2A_5$, $2A_3 + A_5$ of the interaction between the 3-level factors, the treatment effects confounded with the blocks are interaction between more than three factors.

Now, $\operatorname{Ker}\Phi_B^*$ is generated by $A_1 + A_2$, $A_1 + A_4$ and $A_3 + A_5$, whose images by Φ_V^* are $(0,1,0)^{\mathrm{t}}$, $(1,0,0)^{\mathrm{t}}$ and $(0,0,2)^{\mathrm{t}}$. Thus $\operatorname{Ker}\Phi_B^* = \{0, A_3 + A_5, 2A_3 + 2A_5, A_1 + A_2, A_1 + A_2 + A_3 + A_5, A_1 + A_2 + 2A_3 + 2A_5, A_1 + A_4, A_1 + A_4 + A_3 + A_5, A_1 + A_4 + 2A_3 + 2A_5, A_2 + A_4, A_2 + A_4 + A_3 + A_5, A_2 + A_4 + 2A_3 + 2A_5\}$.

The elements which are not in $\operatorname{Ker}\Phi_B^*$ are j-trend orthogonal for any j.

Among the five two-factor interactions which are in $\operatorname{Ker}\Phi_B^*$, the only one having two non-zero coordinates in its image by Φ_V^* is $A_2 + A_4$. This interaction is thus linear trend orthogonal. The other four ones $A_1 + A_2$, $A_1 + A_4$, $A_3 + A_5$, $2A_3 + 2A_5$ are not orthogonal to the linear trend, but they are estimable.

The principal block is generated by the first three columns of Φ, which in multiplicative notation are ab, ad, ce. The order associated with (48) is therefore

$$\{1, ab, ad, bd, ce, abce, acde, bcde, c^2e^2, abc^2e^2, ac^2de^2, bc^2de^2\}$$

To get the orders for the other blocks, we multiply the above sequence by the five elements abd, ce^2, $abdce^2$, c^2e, $abdc^2e$ generated by the last two columns of Φ. Several orders can be selected for the principal block, as in example 2. But the orders for the other blocks must then be defined by the procedure just mentioned.

5 Quantitative factors

The preceding section suggests using the same technique with quantitative factors which are different from the position in time or space. Let A be such a quantitative factor. To use the previous results, we assume that A is the product of several pseudofactors, say $A_1, \dots, A_q$. More precisely, A is the projection from $T = (n_1) \times \cdots \times (n_s)$ onto the group $T_A = (n_1) \times \cdots \times (n_q)$ which sends $\mathbf{t} = (t_1, \dots, t_s)$ onto $\mathbf{a} = (t_1, \dots, t_q)$. The quantitative level associated to $\mathbf{a}$ is then taken as

$$f(\mathbf{a}) = t_1 K_1 + \cdots + t_q K_q \tag{50}$$

The function f can be considered as a vector of $\mathbb{R}^{T_A}$. By orthogonalization for the usual inner product of its successive powers f^0, f^1, … , f^i, … , we get an orthogonal basis g_0, g_1, … , g_i, … of $\mathbb{R}^{T_A}$. The link between this basis and the orthogonal basis of characters of T_A is given by

$$g_i = \sum_{\mathbf{a}^*} \frac{\left\langle g_i, \eta^{\mathbf{a}^*} \right\rangle}{\left\langle \eta^{\mathbf{a}^*}, \eta^{\mathbf{a}^*} \right\rangle} \eta^{\mathbf{a}^*} , \tag{51}$$

$$\eta^{\mathbf{a}^*} = \sum_{i} \frac{\left\langle \eta^{\mathbf{a}^*}, g_i \right\rangle}{\left\langle g_i, g_i \right\rangle} g_i . \tag{52}$$

Let $|\mathbf{a}^*|$ be the number of non-zero coordinates of $\mathbf{a}^* = (t_1^*, \dots, t_q^*)^{\mathrm{t}}$. It follows from proposition 3.1 that $\left\langle g_i, \eta^{\mathbf{a}^*} \right\rangle = 0$ whenever $|\mathbf{a}^*| > i$. Hence the sum in (51) can be restricted to the elements $\mathbf{a}^* \in T_A^*$ satisfying $|\mathbf{a}^*| \leq i$, and the sum in (52) to the indices $i \geq |\mathbf{a}^*|$.

Since A is a surjective morphism from T onto T_A, it is clear that the composite maps $g_0 \circ A$, $g_1 \circ A$, … , $g_i \circ A$, … form an orthogonal basis of the subspace of $\mathbb{C}^T$ generated by the characters $\eta^{\mathbf{a}^*} \circ A = A_1^{t_1^*} \cdots A_q^{t_q^*}$. These maps are called the *polynomial factors of degree* 0, 1, … , *i*, … *in* A and denoted by $pol_0 A$, $pol_1 A$, … , $pol_i A$, … The polynomial factors of degree 1, 2 and 3 are also called the linear, quadratic and cubic factors in A and denoted by $lin\,A$, $quad\,A$ and $cub\,A$. The relations (51) and (52) give

$$pol_i A = g_i \circ A = \sum_{\mathbf{a}^*,\ |\mathbf{a}^*| \leq i} \frac{\left\langle g_i, \eta^{\mathbf{a}^*} \right\rangle}{\left\langle \eta^{\mathbf{a}^*}, \eta^{\mathbf{a}^*} \right\rangle} A_1^{t_1^*} \cdots A_q^{t_q^*} , \tag{53}$$

$$A_1^{t_1^*} \cdots A_q^{t_q^*} = \eta^{\mathbf{a}^*} \circ A = \sum_{i,\ i \geq |\mathbf{a}^*|} \frac{\left\langle \eta^{\mathbf{a}^*}, g_i \right\rangle}{\left\langle g_i, g_i \right\rangle} pol_i A . \tag{54}$$

Let B be a second quantitative factor. We assume that B is the product of the following $p - q$ pseudofactors $A_{q+1}, \dots, A_p$, i.e. that B sends $\mathbf{t} =$

$(t_1, \ldots, t_s)$ onto $\mathbf{b} = (t_{q+1}, \ldots, t_p)$. The associated quantitative levels are defined by a function similar to f, and by orthogonalization of the successive powers of this function, an orthogonal basis $h_0, h_1, \ldots, h_j, \ldots$ of $\mathbb{R}^{T_B}$ is obtained. The polynomial factors of B are defined as those of A : $pol_j B = h_j \circ B$.

The coordinatewise product $pol_i A\ pol_j B$ is then called the *polynomial factor of degree i in A and j in B*. It is not difficult to check that these polynomial factors form an orthogonal basis of the vector space generated by the characters $A_1^{t_1^*} \cdots A_p^{t_p^*}$. Using the bilinearity of the coordinatewise product in $\mathbb{C}^T$, the following identities, where $\mathbf{b}^* = (t_{q+1}^*, \ldots, t_p^*)^{\mathrm{t}}$, can then be deduced from (53) and (54):

$$pol_i A\ pol_j B = \sum_{\substack{\mathbf{a}^*, \mathbf{b}^* \\ |\mathbf{a}^*| \leq i, |\mathbf{b}^*| \leq j}} \frac{\langle g_i, \eta^{\mathbf{a}^*} \rangle}{\langle \eta^{\mathbf{a}^*}, \eta^{\mathbf{a}^*} \rangle} \frac{\langle h_j, \eta^{\mathbf{b}^*} \rangle}{\langle \eta^{\mathbf{b}^*}, \eta^{\mathbf{b}^*} \rangle} A_1^{t_1^*} \cdots A_p^{t_p^*} \tag{55}$$

$$A_1^{t_1^*} \cdots A_p^{t_p^*} = (\eta^{\mathbf{a}^*} \circ A)\ (\eta^{\mathbf{b}^*} \circ B) = \sum_{\substack{i,\ i \geq |\mathbf{a}^*| \\ j,\ j \geq |\mathbf{b}^*|}} \frac{\langle \eta^{\mathbf{a}^*}, g_i \rangle}{\langle g_i, g_i \rangle} \frac{\langle \eta^{\mathbf{b}^*}, h_j \rangle}{\langle h_j, h_j \rangle} pol_i A\ pol_j B \tag{56}$$

With each polynomial factor F on T is associated a linear form $\mathrm{eff}(F)$ of the vector $\boldsymbol{\tau}$ of treatment effect, which is defined by an identity similar to (7) :

$$\mathrm{eff}(F) = \langle \boldsymbol{\tau}, F \rangle / n\ . \tag{57}$$

This linear form is called the *polynomial effect* associated to F. The same terminology and notations are generally used for a polynomial factor and for the corresponding polynomial effect. Thus $\mathrm{eff}(pol_i A\, pol_j B)$ is called the polynomial effect of degree i in A, j in B, and is more concisely denoted by $pol_i A\, pol_j B$.

Taking inner products with $\boldsymbol{\tau}$ in (55) and (56) and dividing by n, we see that these equalities still hold if characters and polynomial factors are replaced by the corresponding effects. These equalities can be used, in conjunction with the group morphism method, to get efficient designs adapted to the quantitative nature of factors A, B.

Suppose indeed that, among the polynomial effects in A and B, only those of degree $\leq d_0$ are of interest, and that those of degree $\geq d_1$, where $d_1 > d_0$, are all zero. From (56), all factorial effects $\mathbf{t}^* = (t_1^*, \ldots, t_p^*, 0, \ldots, 0)$ with $|\mathbf{t}^*| \geq d_1$ are zero. Using this fact and other possible hypothesis on factorial effects, we can select a block fractional design in which factorial effects $\mathbf{t}^* = (t_1^*, \ldots, t_p^*, 0, \ldots, 0)$ satisfying $|\mathbf{t}^*| \leq d_0$ are unconfounded, hence orthogonally estimated with factor efficiency 1. By (55), the polynomial effects of degree $\leq d_0$ can then also be estimated with factor efficiency 1.

Example 4 . In a study on the dissociation of goat milk casein micelles, three factors are varied : pH (8 levels), Temperature (4 levels) and added Calcium (2 levels). The choice of 8 levels for pH is motivated by the shape of the corresponding curve for cow milk which had been previously obtained by Dalgleish and Law (1988) (fig 1).

The factors pH and T (temperature) are decomposed into products of three and two pseudofactors respectively, and the association with quantitative levels is chosen as indicated by table 3. Every factorial effect defined with d symbols is then a linear combination of polynomial effects of degree at least d, and any polynomial effect of degree smaller or equal to d is a linear combination of factorial effects involving at most d symbols.

pH	pH_1	pH_2	pH_3
-7	1	1	1
-5	-1	1	1
-3	1	-1	1
-1	-1	-1	1
1	1	1	-1
3	-1	1	-1
5	1	-1	-1
7	-1	-1	-1

T	T_1	T_2
-3	1	1
-1	-1	1
1	1	-1
3	-1	-1

Table 3: Decomposition into pseudofactors of pH and T

Consider now the fraction defined by $pH_1 pH_2 pH_3 T_1 T_2 C = 1$. If polynomial effects of degree 4 are assumed to be zero, then all factorial effects involving at least 4 symbols are zero. Hence all factorial effects involving two symbols at most are estimable with factor efficiency 1 and so are the polynomial effects of degree smaller than or equal to 2.

Figure 1: effect of pH and T on the dissociation of casein micelles for cow milk

This example is inspired by a greater design, a half fraction of a $8 \times 4 \times 2 \times 2 \times 2$ in 8 blocks of size 16, established for Anne Jobert Vesperini and J.P. Quiblier (Laboratoire de recherche de Technologie laitière. INRA Rennes). The blocks were constituted by the samples coming from the same milking. A detailed description of this design is available (Kobilinsky 1990b).

6 Minimum cost run orders

We consider a design defined by a map $\phi : U \to T$ satisfying (14) and search for run orders with minimal cost. If $\mathbf{u}_1, \ldots, \mathbf{u}_N$ is an ordered sequence of the runs in U, the corresponding cost is assumed to be:

$$\sum_{i=2}^{N} C\Big(\phi(\mathbf{u}_{i+1}) - \phi(\mathbf{u}_i)\Big) = \sum_{i=2}^{N} C \circ \Phi(\mathbf{u}_{i+1} - \mathbf{u}_i) \qquad (58)$$

where the cost function C satisfies:

$$C(\mathbf{t}) = C(-\mathbf{t}) \; . \qquad (59)$$

Here $C \circ \phi(\mathbf{u})$ is the cost of the changes of levels between successive runs whose difference is $\mathbf{u}$. For simplicity, we call it the *cost of* $\mathbf{u}$ and put

$$c(\mathbf{u}) = C \circ \Phi(\mathbf{u}) \ . \tag{60}$$

A reasonable choice for C is for instance

$$C(\mathbf{t}) = \sum_{\substack{j \\ t_j \neq 0}} \delta_j, \tag{61}$$

which assumes that the cost of a change of level of factor j is δ_j, where $\delta_j \geq 0$.

We now give a general process to get run orders of minimal cost.

Define a system $\mathbf{z}_1, \ldots, \mathbf{z}_h$ of generators of U by choosing

- $\mathbf{z}_1$ among the elements of minimum cost of $U \setminus \{0\}$,
- $\mathbf{z}_2$ among the elements of minimum cost of $U \setminus \langle \mathbf{z}_1 \rangle$,

...

- $\mathbf{z}_i$ among the elements of minimum cost of $U \setminus \langle \mathbf{z}_1, \ldots, \mathbf{z}_{i-1} \rangle$,

...

- $\mathbf{z}_h$ among the elements of minimum cost of $U \setminus \langle \mathbf{z}_1, \ldots, \mathbf{z}_{h-1} \rangle$. The process stops here because

$$\langle \mathbf{z}_1, \ldots, \mathbf{z}_h \rangle = U \ .$$

Here, $\langle \mathbf{z}_1, \ldots, \mathbf{z}_{i-1} \rangle$ refers to the subgroup of U generated by $\mathbf{z}_1, \ldots, \mathbf{z}_{i-1}$, while $U \setminus \langle \mathbf{z}_1, \ldots, \mathbf{z}_{i-1} \rangle$ is the set of elements of U which do not belong to this subgroup.

For every $i = 1, \ldots, h$, let o_i be the period of $\mathbf{z}_i$ in the quotient group $U / \langle \mathbf{z}_1, \ldots, \mathbf{z}_{i-1} \rangle$, i.e. the smallest integer satisfying $o_i \mathbf{z}_i \in \langle \mathbf{z}_1, \ldots, \mathbf{z}_{i-1} \rangle$.

The elements of U have a unique expression as linear combinations $a_1 \mathbf{z}_1 + \ldots + a_h \mathbf{z}_h$, with integer coefficients a_i satisfying $0 \leq a_i < o_i$. They can be ordered with a minimal cost by a method (called the foldover method by Cheng 1990 in symmetrical cases) which we describe first in Table 4 for the case $h = 2$.

The rows of that table are the cosets of $\langle \mathbf{z}_1 \rangle$ in $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle$. The cost to go from one element to the next in each row is the minimal cost $c(\mathbf{z}_1)$, while the cost to go from the end of a row to the beginning of the next one is the minimal cost $c(\mathbf{z}_2)$ for jumping from one coset of $\langle \mathbf{z}_1 \rangle$ to another. Therefore, the cost for one row is $(o_1 - 1)c(\mathbf{z}_1)$ and the cost for the whole sequence $\mathcal{U}_2$ of $o_1 o_2$ elements of the table is :

$$c(\mathcal{U}_2) = o_2(o_1 - 1)c(\mathbf{z}_1) + (o_2 - 1)c(\mathbf{z}_2) \ . \tag{62}$$

0	$\longrightarrow$	$\mathbf{z}_1$	$\longrightarrow \ldots \longrightarrow$	$(o_1 - 1)\mathbf{z}_1$
$(o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2$	$\longrightarrow$	$\mathbf{z}_1 + (o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2$	$\longrightarrow \ldots \longrightarrow$	$(o_1 - 1)\mathbf{z}_1 + (o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2$
$2[(o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2]$	$\longrightarrow$	$\mathbf{z}_1 + 2[(o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2]$	$\longrightarrow \ldots \longrightarrow$	$(o_1 - 1)\mathbf{z}_1 + 2[(o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2]$
		$\vdots$		
$(o_2 - 1)[(o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2]$	$\longrightarrow$	$\mathbf{z}_1 + (o_2 - 1)[(o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2]$	$\longrightarrow \ldots \longrightarrow$	$(o_1 - 1)\mathbf{z}_1 + (o_2 - 1)[(o_1 - 1)\mathbf{z}_1 + \mathbf{z}_2]$

Table 4: Minimum cost order in the case $h = 2$

The last element of this table is

$$\boldsymbol{\omega}_2 = o_2(o_1 - 1)\mathbf{z}_1 + (o_2 - 1)\mathbf{z}_2 \ .$$

To get the foldover order in the case $h = 3$, the quantities $\mathbf{x}_3, \ldots, (o_3 - 1)\mathbf{x}_3$, where

$$\mathbf{x}_3 = \boldsymbol{\omega}_2 + \mathbf{z}_3 = o_2(o_1 - 1)\mathbf{z}_1 + (o_2 - 1)\mathbf{z}_2 + \mathbf{z}_3,$$

are successively added to the elements of Table 4. The last element is then:

$$\boldsymbol{\omega}_3 = (o_3 - 1)\mathbf{x}_3 + \boldsymbol{\omega}_2 = o_3\boldsymbol{\omega}_2 + (o_3 - 1)\mathbf{z}_3 = o_3 o_2(o_1 - 1)\mathbf{z}_1 + o_3(o_2 - 1)\mathbf{z}_2 + (o_3 - 1)\mathbf{z}_3 \ .$$

It is not difficult to go on in the same manner. Let $\mathcal{U}_i$ be the ordered sequence of the elements of $\langle \mathbf{z}_1, \ldots, \mathbf{z}_i \rangle$ obtained at step i, and let $\boldsymbol{\omega}_i$ be the last element of $\mathcal{U}_i$.

The sequence $\mathcal{U}_{i+1}$ is defined by:

$$\mathcal{U}_{i+1} = (\mathcal{U}_i, \mathcal{U}_i + \mathbf{x}_{i+1}, \ldots, \mathcal{U}_i + (o_{i+1} - 1)\mathbf{x}_{i+1}) \tag{63}$$

where

$$\mathbf{x}_{i+1} = \boldsymbol{\omega}_i + \mathbf{z}_{i+1} \ . \tag{64}$$

We now prove by induction that :

$$\boldsymbol{\omega}_i = \sum_{j=1}^{i} \left(\prod_{k=j+1}^{i} o_k \right) (o_j - 1)\mathbf{z}_j \tag{65}$$

(where $\omega_0 = 0$ and $\prod_{k=i+1}^{i} o_k = 1$ by definition). Indeed if (65) is true for i, the last element of $\mathcal{U}_{i+1}$ is

$$\boldsymbol{\omega}_{i+1} = (o_{i+1} - 1)\mathbf{x}_{i+1} + \boldsymbol{\omega}_i = o_{i+1}\boldsymbol{\omega}_i + (o_{i+1} - 1)\mathbf{z}_{i+1} = \sum_{j=1}^{i} \left(\prod_{k=j+1}^{i+1} o_k \right) (o_j - 1)\mathbf{z}_j + (o_{i+1} - 1)\mathbf{z}_{i+}$$

and (65) is valid for $i + 1$.

By a similar induction, the cost $c(\mathcal{U}_i)$ of the sequence $\mathcal{U}_i$ is found to be :

$$c(\mathcal{U}_i) = \sum_{j=1}^{i} \left(\prod_{k=j+1}^{i} o_k \right) (o_j - 1)c(\mathbf{z}_j) \ . \tag{66}$$

Proposition 6.1 *For any ordered sequence* $(\mathbf{u}_1, \ldots, \mathbf{u}_N)$ *of the runs in* U*, the corresponding cost is greater or equal to the cost* $c(\mathcal{U}_h)$ *of the last sequence obtained by the recurrent process* **(63)***.*

Proof. For every j, the sequence $(\mathbf{u}_1, \ldots, \mathbf{u}_N)$ must go through each of the $o_j \cdots o_h$ cosets of $\langle \mathbf{z}_1, \ldots, \mathbf{z}_{j-1} \rangle$ and therefore includes at least $o_j \cdots o_h - 1$ jumps from one coset to another, i.e. consecutive pairs $(\mathbf{u}_i, \mathbf{u}_{i+1})$ such that $\mathbf{u}_{i+1} - \mathbf{u}_i \notin \langle \mathbf{z}_1, \ldots, \mathbf{z}_{j-1} \rangle$. It is therefore possible to constitute successive disjoints sets $\mathcal{T}_h, \ \ldots, \ \mathcal{T}_1$ of couples such that :

$$| \mathcal{T}_j | = o_j \cdots o_h - o_{j+1} \cdots o_h = \left(\prod_{k=j+1}^{h} o_h \right) (o_j - 1)$$

and

$$(\mathbf{u}_i, \mathbf{u}_{i+1}) \in \mathcal{T}_j \Rightarrow \mathbf{u}_{i+1} - \mathbf{u}_i \notin \langle \mathbf{z}_1, \ldots , \mathbf{z}_{j-1} \rangle \ .$$

It follows from the choice of $\mathbf{z}_j$ that if $(\mathbf{u}_i, \mathbf{u}_{i+1}) \in \mathcal{T}_j$, the cost of the jump from $\mathbf{u}_i$ to $\mathbf{u}_{i+1}$ is at least $c(\mathbf{z}_j)$. The cost of the sequence $(\mathbf{u}_1, \ldots , \mathbf{u}_N)$ is therefore greater than or equal to $\sum_{j=1}^{h} | \mathcal{T}_j | c(\mathbf{z}_j) = c(\mathcal{U}_h)$ $\Box$.

Remarks :

- The elements of U are the linear combinations $\mathbf{u} = b_1\mathbf{x}_1 + \cdots + b_h\mathbf{x}_h$, where the generators $\mathbf{x}_1, \ldots, \mathbf{x}_h$ are defined by (64), (65) and the coefficients are integers satisfying $0 \leq \mathbf{b}_i < o_i$. The position number $f(\mathbf{u})$ of such an element in the sequence $\mathcal{U}_h$ is :

$$f(\mathbf{u}) = b_1 + o_1\, b_2 + o_1 o_2\, b_3 + \cdots + \left(\prod_{k=1}^{h-1} o_k\right) b_h \ . \tag{67}$$

  The order of $\mathcal{U}_h$ thus corresponds to a natural order of enumeration of the h-plets $(b_1, \ldots, b_h)$.

- Conversely, let $\mathbf{x}_1, \ldots, \mathbf{x}_h$ be generators of U and define o_i to be the period of $\mathbf{x}_i$ in the quotient group $U/\langle \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\rangle$. The elements of U are the linear combinations $\mathbf{u} = b_1\mathbf{x}_1 + \cdots + b_h\mathbf{x}_h$ with integer coefficients b_i satisfying $0 \leq b_i < o_i$. Assume U is ordered according to the value of the function f given by (67), that is with the natural order 0, $\mathbf{x}_1, \ldots, (o_1 - 1)\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_2 + \mathbf{x}_1, \ldots, \mathbf{x}_2 + (o_1 - 1)\mathbf{x}_1, \ldots,$ $(o_2 - 1)\mathbf{x}_2 + (o_1 - 1)\mathbf{x}_1$, $\mathbf{x}_3, \ldots, \sum_{j=1}^{h}(o_j - 1)\mathbf{x}_j$. Let

$$\mathbf{z}_i = \mathbf{x}_i - \sum_{j=1}^{i-1}(o_j - 1)\mathbf{x}_j \tag{68}$$

  Then, the cost of the sequence is given by (66) for $i = h$.

- Assume that the periods o_j are equal to a given period o and that $o\mathbf{z}_j = 0$ for every j. In that case, it follows from (64) and (65) that for $i \geq 1$

$$\mathbf{x}_{i+1} = (o - 1)\mathbf{z}_i + \mathbf{z}_{i+1} \ . \tag{69}$$

  Recall that $\mathbf{x}_1 = \mathbf{z}_1$.

Example 5 . $2\times2\times2\times3$ in 2 blocks of size $2\times2\times3$, with a minimum number of level changes in each block

The cost $C(\mathbf{t})$ is defined by (61) with all δ_i equal to 1. We let A, B, C, D be the four factors and confound ABC with the blocks. The principal block, defined by $ABC = 1$, is

$$\{1, ab, ac, bc, d, abd, acd, bcd, d^2, abd^2, acd^2, bcd^2\}$$

To order it with a minimum number of level changes, we select $\mathbf{z}_1$ to be d or d^2, then $\mathbf{z}_2$ and $\mathbf{z}_3$ to be ab, ac or bc. A possible order, obtained when $\mathbf{z}_2 = ab$, $\mathbf{z}_3 = ac$, is for instance

$$\{1, d, d^2, abd^2, ab, abd, bcd, bcd^2, bc, ac, acd, acd^2\}$$

Many other orders are possible, depending on the choice of $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$. Moreover, the sequence can be multiplied by any of its elements without changing the cost. Multiplying these orders by an element, say a, of the second block, one can get orders with a minimum number of level changes for the second block. In each of these orders however, the two level factors remain constants on series of three consecutive units, which can raise serious statistical problems if there are correlations between successive unit effects within blocks.

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