Juxtaposition of regular factorial designs and the complex linear model

A. Kobilinsky, H.Monod

INRA Versailles. Laboratoire de Biométrie. France.
1995, *Scand. J. Statist.* **22**

ABSTRACT The use of irreducible characters of a commutative group to get a complex reparametrization is reviewed. It is shown that complex parameters can be handled very similarly to real ones and that, for non-orthogonal designs obtained by assembling regular fractions or block fractional designs, they simplify obtaining factor efficiencies considerably and thus give simple clues for the construction.

Key words: abelian group, factorial design, partial confounding, fractional designs, cyclic design, efficiency factor.

1 Introduction

Many useful designs are created by juxtaposing several subdesigns obtained by the group morphism method described in Kobilinsky & Monod (1991). Lattices are well known examples of such juxtapositions in which the subdesigns are the different replicates. On a single replicate it is impossible to avoid the confounding with the blocks of some treatment contrasts. However, as explained for instance in Kempthorne (1952), the confounding on the whole design is only partial because the treatment effects confounded differ from one replicate to the other.

It was shown in Kobilinsky (1990) that the use of the complex reparametrization associated with the characters of the group of treatments (Chakravarti 1976, Bailey 1982) leads to drastic simplifications in the analysis and construction of these designs. A practically important application was the study of a new scheme giving a useful method of blocking for single replicate or even fractional factorial designs.

The use of parameters in the field $\mathbf{C}$ of complex numbers raises some questions. Is it possible to deal with them exactly as in the real case? The answer is yes provided these complex parameters, and the corresponding columns of the incidence matrix X are conjugated by pairs. We recall here the main results about this kind of "*complex*" linear model and apply them to a number of situations: the scheme just mentioned, semi-regular fractional designs generalizing in a natural way the three-quarter fraction of P.W.M. John (1962) and other block factorial symmetrical or asymmetrical designs for 2 and 3-level factors. Explicit simple formulae for factorial efficiencies are in some cases easy to derive and can help in choosing a good design.

2 Characters of the group of treatments

We suppose as in Kobilinsky & Monod (1991) that the treatments are the $n = n_1 \times \cdots \times n_s$ elements of a product group $T = (n_1) \times \cdots \times (n_s)$ where (n_i), the additive cyclic group of order n_i, is used to represent the levels of the i^{th} factor A_i. If M is a common multiple of n_1, ..., n_s, all these levels can be embedded into the cyclic group of order M, which is represented either as (M), the additive group of integers modulo M, or as R_M the multiplicative group of M^{th} roots of unity in $\mathbf{C}$. The embedding associates with level t_i in (n_i) either $t_i M/n_i$ in (M), or $\exp(t_i\, 2\pi \mathrm{i}/n_i)$ in R_M. The factors are then considered as mappings from T into (M) or R_M. Thus if $\mathbf{t} = (t_1, \ldots, t_s)$ is a treatment in T, $A_i(\mathbf{t})$ is the corresponding level of factor A_i in (M) or R_M:

$$A_i(\mathbf{t}) = \frac{M}{n_i} t_i \quad \text{in } (M), \text{ or} \tag{1}$$

$$A_i(\mathbf{t}) = \exp\left(\frac{2\pi \mathrm{i}}{n_i} t_i\right) \quad \text{in } R_M. \tag{2}$$

A character [1] A of T is a group morphism from T into the cyclic group of order M, that is in (M) or R_M. The operation of this group induces a similar one on the characters. Thus the sum $A + B$ (resp. product AB) of two characters is defined by

$$(A + B)(\mathbf{t}) = A(\mathbf{t}) + B(\mathbf{t}) \; . \tag{3}$$

$$\text{resp. } \; (AB)(\mathbf{t}) = A(\mathbf{t})B(\mathbf{t}) \tag{4}$$

Under this operation, the set of characters is an abelian group called the dual group of T and denoted by T^*.

Proposition 2.1 *The mapping* $(a_1, \ldots, a_s) \mapsto a_1 A_1 + \cdots + a_s A_s$ *(resp.* $A_1^{a_1} \cdots A_s^{a_s}$*) is an isomorphism from* $(n_1) \times \cdots \times (n_s)$ *onto* T^*.

(See El Mossadeq, Kobilinsky 1992 for a proof). The dual T^* can therefore be identified with the product group $(n_1) \times \cdots \times (n_s)$. For each $\mathbf{a} = (a_1, \ldots, a_s)$ in T^* and $\mathbf{t} = (t_1, \ldots, t_s)$ in T, we denote by $[\mathbf{a}, \mathbf{t}]$ the image of $\mathbf{t}$ by $a_1 A_1 + \cdots + a_s A_s$, which is

$$[\mathbf{a}, \mathbf{t}] = \frac{M}{n_1} a_1 t_1 + \cdots + \frac{M}{n_s} a_s t_s \tag{5}$$

The corresponding element in R_M is $\eta^{[\mathbf{a},\mathbf{t}]}$ where η is the primitive M^{th} root of unity defined by

$$\eta = \exp(2\pi \mathrm{i}/M) \; . \tag{6}$$

The morphism $A = A_1^{a_1} \cdots A_s^{a_s}$ associated with $\mathbf{a}$, which sends $\mathbf{t}$ on $\eta^{[\mathbf{a},\mathbf{t}]}$ is denoted by $\eta^{\mathbf{a}}$. Thus

$$\eta^{\mathbf{a}}(\mathbf{t}) = \eta^{[\mathbf{a},\mathbf{t}]} \tag{7}$$

[1] the exact terminology is *irreducible character*. The qualifier irreducible will, however, be dropped since all characters used in this text are irreducible.

and the mapping $\mathbf{a} \mapsto \eta^{\mathbf{a}}$ is by Proposition 2.1 an isomorphism from the additive group $(n_1) \times \cdots \times (n_s)$ onto the multiplicative group of characters of T.

Proposition 2.1 shows that any character can be generated in a unique way from the factors $A_1, \ldots, A_s$, which will therefore be called the *"basic characters"* of T. Each character A can then be considered as a new factor derived from the basic ones, and it will sometimes be referred to as the *pseudofactor* or *factor* A.

The *multiplicative* characters have a very important property. They form an orthogonal basis of $\mathbf{C}^T$ for the usual inner product [2] and have the same square norm n. Thus if A and B are characters

$$\langle A, B \rangle = \sum_{\mathbf{t} \in T} A(\mathbf{t}) \overline{B(\mathbf{t})} = 0 \tag{8}$$

$$\langle A, A \rangle = \sum_{\mathbf{t} \in T} |A(\mathbf{t})|^2 = n \tag{9}$$

3 Canonical parameters

Let $\tau = (\tau(\mathbf{t}))_{\mathbf{t} \in T}$ be the $n \times 1$ vector of treatment effects. The decomposition of this vector on the orthogonal basis of characters of T is

$$\tau = \sum_{A} e(A) A \tag{10}$$

where the coefficients $e(A)$ are

$$e(A) = \langle \tau, A \rangle / n = \sum_{\mathbf{t} \in T} \tau(\mathbf{t}) \overline{A(\mathbf{t})} / n \tag{11}$$

The characters have here their multiplicative form with values in R_M. The linear form $e(A)$ is called the *canonical parameter*, or *factorial effect*, or more simply *contrast* associated with A. If the context makes things clear, no distinction is made between a character and its associated linear form. Thus $e(A)$ is referred to as the parameter, effect or contrast A and the dual T^* is sometimes viewed as the group of all factorial effects. In this terminology, $\mathbf{a}$ may be substituted for $A = \eta^{\mathbf{a}}$. We then speak of the contrast $\mathbf{a}$ and write $e(\mathbf{a})$ instead of $e(\eta^{\mathbf{a}})$.

It is known and easy to prove that the factorial effects associated with the powers $A_i, \ldots, A_i^{n_i - 1}$ span the $n_i - 1$ degrees of freedom of the main effect of factor A_i. Similarly the canonical parameters associated with the $(n_{i_1} - 1) \cdots (n_{i_k} - 1)$ characters $A_{i_1}^{a_{i_1}} \cdots A_{i_k}^{a_{i_k}}$, where $a_{i_1}, \ldots, a_{i_k}$ are all different from 0, span the interaction between the k factors A_{i_1}, $\ldots$, A_{i_k}.

It is often assumed that some interactions are zero. It is equivalent to assume that the canonical parameters spanning these interactions are zero. If this is the case, the sum in (10) can be restricted to the subset S^* of characters A associated with non-zero canonical parameters. Note that S^* is stable by conjugation since when $A \neq \overline{A}$, the two

[2] $\mathbf{C}^T$ is the set of vectors $(x_t)_{t \in T}$ indexed by the elements of T and with coordinates in $\mathbf{C}$. Such a vector can be identified with the mapping $t \mapsto x_t$.

conjugated parameters $e(A)$ and $e(\overline{A})$ both belong to the same interaction or main effect and are thus simultaneously assumed or not to be zero. With these assumptions, the domain of definition $\mathcal{T}$ of the vector of treatment effects is the subspace of vectors τ in $\mathbf{R}^T$ satisfying $e(A) = 0$ for $A \notin S^*$:

$$\mathcal{T} = \{\tau \in \mathbf{R}^T / \langle \tau, A \rangle = 0 \quad \text{for } A \notin S^*\} \tag{12}$$

Since $e\left(\overline{A}\right) = \overline{e(A)}$ for every A in S^*, the vector $\theta = (e(A))_{A \in S^*}$ of non-zero canonical parameters does not vary freely in $\mathbf{C}^{S^*}$. It belongs to the *real* subspace $\mathcal{D}$ of vectors β in $\mathbf{C}^{S^*}$ satisfying $\beta\left(\overline{A}\right) = \overline{\beta(A)}$:

$$\mathcal{D} = \{\beta \in \mathbf{C}^{S^*} / \beta\left(\overline{A}\right) = \overline{\beta(A)} \quad \text{for } A \in S^*\}. \tag{13}$$

Proposition 3.1 *The two real vector spaces $\mathcal{T}$ and $\mathcal{D}$ are isomorphic. The isomorphism is the mapping $\tau \mapsto \theta = (e(A))_{A \in S^*}$ defined by* (10) *or* (11) .

The next proposition shows that the *real* linear forms $\tau \mapsto \langle \tau, a \rangle$ on $\mathcal{T}$ $(a \in \mathbf{R}^T)$ are the linear combinations $\sum_A \lambda(A) e(A)$ with $\lambda \in \mathcal{D}$.

Proposition 3.2 *A linear form $\sum_A \lambda(A) e(A)$ of τ is real if and only if λ is in $\mathcal{D}$. Moreover, the mapping which to each λ in $\mathcal{D}$ associates the real linear form $\sum_A \lambda(A) e(A)$ of τ is a real vector space isomorphism between $\mathcal{D}$ and $\mathcal{T}^*$.*

Proof. If $\sum_A \lambda(A) e(A)$ is real for every τ in $\mathcal{T}$, $\lambda(A) + \lambda(\overline{A})$ and $\lambda(A)\mathrm{i} - \lambda(\overline{A})\mathrm{i}$ are real (take $e(A) = 1$ or $e(A) = \mathrm{i}$ and $e(B) = 0$ for $B \neq A, \overline{A}$). This implies that $\lambda(\overline{A}) = \overline{\lambda(A)}$. It can be similarly shown that if $\sum_A \lambda(A) e(A) = 0$ for every τ, $\lambda = 0$.

It is then clear that $\lambda \mapsto \sum_A \lambda(A) e(A) = \left\langle \tau, \sum_A \overline{\lambda(A)} A / n \right\rangle$ is an injective linear mapping from $\mathcal{D}$ into $\mathcal{T}^*$. It is surjective since for $a \in \mathbf{R}^T$, $\langle \tau, a \rangle = \sum_A \langle A, a \rangle e(A)$ and $\left\langle \overline{A}, a \right\rangle = \overline{\langle A, a \rangle}$. □

Let $\phi : U \to T$ be the mapping of allocation of treatments to units in a particular design and suppose that the expectation of the response $y(\mathbf{u})$ on unit $\mathbf{u}$ depends only on the corresponding treatment $\phi(\mathbf{u})$:

$$E\left(y(\mathbf{u})\right) = \tau\left(\phi(\mathbf{u})\right) \tag{14}$$

Then the expectation of the vector of responses $y = (y(\mathbf{u}))$ is $\tau \circ \phi$. Using (10) it can be expressed in function of the canonical parameters:

$$E(y) = \sum_A e(A) A \circ \phi \, . \tag{15}$$

If X is the matrix with the vectors $A \circ \phi$ in columns, we thus have

$$E(y) = X\theta \quad \text{with } \theta \in \mathcal{D}. \tag{16}$$

The parameters in θ are either real or conjugated by pair, and so are the corresponding columns of X. Conjugated parameters can be recombined to get a model with real parameters, as done in Kobilinsky (1985). However, as pointed out in Kobilinsky (1990) it is better to work directly with the complex canonical parameters. The normal equations are

$$X^*X\hat{\theta} = X^*y \; , \tag{17}$$

with the conjugate transpose X^* instead of X^{t}. The classical definitions and results for the linear model can be extended to this *complex* model. For instance an estimable function can be defined as follows.

Definition 3.1 (Estimable function) *A linear form on $\mathcal{T}$ (or on $\mathcal{D}$) is* estimable *if it is of the form $\langle E(y), a\rangle$ for a given a in $\mathbf{C}^U$. An* estimable function *is an estimable linear form.*

Usually we are interested in real linear forms for which $a \in \mathbf{R}^U$. But it can also be interesting to consider complex linear forms like the canonical parameters. Since

$$\langle E(y), \overline{a}\rangle = \overline{\langle E(y), a\rangle}$$

the conjugated of a complex estimable function is also an estimable function, and so are its real and imaginary parts. Note that conversely, a complex function whose real and imaginary parts are both estimable is itself estimable. However, in Section 4 we shall give an example of a canonical parameter whose real part is estimable but which is not itself estimable.

We study more thoroughly complex models like (16) in Section 6, but first illustrate by some examples the interest of this canonical complex reparametrization. For designs whose construction involves abelian groups, it leads to a block diagonal matrix X^*X easier to manipulate than any normal equation matrix linked to a real parametrization.

4 The group morphism construction method

In this method, experimental units are represented by a product group $U = (m_1) \times \cdots \times (m_r)$ and levels of block factors, if any, are also given by the elements of a group $V = (p_1) \times \cdots \times (p_q)$. The mappings $\phi : U \to T$ and $\psi : U \to V$ defining the treatment $\mathbf{t} = \phi(\mathbf{u})$ and blocks $\mathbf{v} = \psi(\mathbf{u})$ on unit $\mathbf{u}$ are of the form

$$\phi(\mathbf{u}) = \mathbf{t}_0 + \Phi\mathbf{u} \qquad \psi(\mathbf{u}) = \mathbf{v}_0 + \Psi\mathbf{u} \tag{18}$$

where $\Phi : U \to T$, $\Psi : U \to V$ are group morphisms and $\mathbf{t}_0 \in T$, $\mathbf{v}_0 \in V$ fixed elements.

Definition 4.1 (GMGD) *The designs defined by* (18) *are called Group Morphism Generated Designs, or more briefly GMG Designs or GMGD.*

If there is no block factor and if Φ is injective, the design defined by (18) is also called a *regular fraction.* Thus a regular fraction is a coset $\mathbf{t}_0 + \mathrm{Im}\,\Phi$ of a subgroup $\mathrm{Im}\,\Phi$ of the group of treatments.

The expected response on unit $\mathbf{u}$ is supposed to be the sum of the corresponding treatment and block effects:

$$E\left(y(\mathbf{u})\right) = \tau(\phi(\mathbf{u})) + \zeta(\psi(\mathbf{u}))$$

The expectation of the vector y of responses on the $m = m_1 \times \cdots \times m_r$ units of U is then

$$E(y) = \tau \circ \phi + \zeta \circ \psi \tag{19}$$

Using (10) and the similar decomposition of block effects

$$\zeta = \sum_B e(B) B \tag{20}$$

we get

$$E(y) = \sum_A e(A) A \circ \phi + \sum_B e(B) B \circ \psi \tag{21}$$

For each $\mathbf{u}$, $A \circ \phi(\mathbf{u}) = A(\mathbf{t}_0 + \Phi\mathbf{u}) = A(\mathbf{t}_0) A \circ \Phi(\mathbf{u})$, hence $A \circ \phi = A(\mathbf{t}_0) A \circ \Phi$ and similarly $B \circ \psi = B(\mathbf{v}_0) B \circ \Psi$. Thus

$$E(y) = \sum_A A(\mathbf{t}_0) e(A) A \circ \Phi + \sum_B B(\mathbf{v}_0) e(B) B \circ \Psi \, . \tag{22}$$

Since A, B, Φ, Ψ are morphisms, so are the composition maps $A \circ \Phi$ and $B \circ \Psi$ which are therefore characters of U. Two of them are either orthogonal or collinear. The grouping of collinear columns then gives

$$E(y) = \sum_C \gamma(C) C \tag{23}$$

where for each $C \in U^*$

$$\gamma(C) = \sum_{A : A \circ \Phi = C} A(\mathbf{t}_0) e(A) + \sum_{B : B \circ \Psi = C} B(\mathbf{v}_0) e(B) \, . \tag{24}$$

The above summations can be restricted to *non-zero effects*, that is to effects which are not zero by hypothesis. Note that $\gamma(C)$ is assumed to be zero iff all effects $e(A)$ and $e(B)$ appearing in (24) are assumed to be zero.

Definition 4.2 Confounded parameters. *The canonical parameters $e(A)$ and $e(B)$ appearing in* (24) *are said to be* confounded *or* aliased *with C on U. The same is said of the corresponding characters. The set of confounded treatment parameters $e(A)$ is also called an* alias set.

The reference to U is often implied by the context and not explicitly mentioned.

Thus, the effects (characters) of T^* or V^* confounded with $C \in U^*$ are those which have C as image by the mappings $\Phi^* : A \mapsto A \circ \Phi$ and $\Psi^* : B \mapsto B \circ \Psi$. It is easy to check that these mappings, known as the *"duals"* of Φ and Ψ respectively, are group morphisms from T^* and V^* into U^*. Therefore if we know a treatment effect A confounded with C, the other ones can be obtained as the non-zero effects in the coset $A \operatorname{Ker} \Phi^*$ ($A + \operatorname{Ker} \Phi^*$ in additive notations).

Estimable functions

The notations of section 3 are now adjusted to take the block effects into account. Thus $\mathcal{T}$ is the subspace of definition of the whole vector $(\tau^t, \zeta^t)^t$ of treatment and block effects. The set S^* includes all treatment and block characters associated with non-zero canonical parameters and $\theta = (e(A))_{A\in S^*}$ is the corresponding vector of parameters, which varies in the real subspace $\mathcal{D}$ defined by (13). The mapping $(\tau^t, \zeta^t)^t \mapsto \theta$ defined by (10) and (20) is a real vector space isomorphism between $\mathcal{T}$ and $\mathcal{D}$. Each real linear form on $\mathcal{T}$ can be written as a sum $\sum_{A\in S^*} \lambda(A)e(A)$ with λ uniquely defined in $\mathcal{D}$.

The following proposition follows immediately from (23).

Proposition 4.1 *The estimable functions are the functions $\sum_C \lambda(C)\gamma(C)$ generated by the non-zero linear forms $\gamma(C)$.*

As a consequence an effect $e(A)$ is estimable, and so are its real and imaginary parts, iff it is not confounded with any other effect. Note however that, even when $e(A)$ is not estimable, its real part or some other linear combination of $e(A)$ and $e(\overline{A})$ can be estimable. Indeed, let A be a four-level factor such that $A \circ \phi = C$ and suppose C is real, satisfying $C^2 = \mathbf{1}$. Then $A^3 \circ \phi = C^3 = C$. Hence A and $\overline{A} = A^3$ are confounded with C. If they are the only non-zero effects confounded with C, $\gamma(C) = e(A) + e(\overline{A})$ and thus $\Re\, e(A)$ is estimable, but $e(A)$ is not.

Definition 4.3 (CEF) *The non-zero combinations of confounded canonical parameters $\gamma(C)$ defined by* (24) *are called the Canonical Estimable Functions on U.*

Estimation of the canonical estimable functions

The observations are assumed to be uncorrelated and of common variance σ^2. The model can therefore be written

$$E(y) = Z\gamma \qquad \text{var}(y) = \sigma^2\mathbf{I}\ . \tag{25}$$

The columns of the matrix Z are the characters C of U associated with non-zero canonical estimable functions $\gamma(C)$. The vector γ of non-zero effects belongs to the subspace $\mathcal{D}_\gamma$ of vectors whose coordinates of conjugated indices in U^* are conjugated.

The orthogonality properties of the characters imply

$$Z^*Z = m\mathbf{I}\ . \tag{26}$$

Hence the least squares estimate of γ and its variance-covariance matrix are

$$\hat{\gamma} = \frac{1}{m}Z^*y \qquad \text{var}\,(\hat{\gamma}) = \frac{\sigma^2}{m}\mathbf{I} \tag{27}$$

The corresponding coordinatewise expression is

$$\hat{\gamma}(C) = \langle y, C\rangle / m \quad \text{var}\,(\hat{\gamma}(C)) = \sigma^2/m \quad \text{cov}\Big(\hat{\gamma}(C), \hat{\gamma}(C')\Big) = 0 \quad \text{for } C \neq C' \tag{28}$$

The definition of variances and covariances for random variables with values in $\mathbf{C}$ is straightforward:

$$\begin{aligned}\operatorname{cov}(x_1, x_2) &= E\left[\left(x_1 - E(x_1)\right)\left(\overline{x_2 - E(x_2)}\right)\right], &(29)\\ \operatorname{var}(x) &= \operatorname{cov}(x, x) . &(30)\end{aligned}$$

The covariance is an Hermitian form. Hence

$$\begin{aligned}\operatorname{cov}\left(\hat{\gamma}(C), \hat{\gamma}(C')\right) &= \operatorname{cov}\left(\sum_{\mathbf{u}} \overline{C}(\mathbf{u}) y(\mathbf{u})/m, \sum_{\mathbf{u}} \overline{C'}(\mathbf{u}) y(\mathbf{u})/m\right)\\ &= \sum_{\mathbf{u}} \overline{C}(\mathbf{u}) C'(\mathbf{u}) \operatorname{var}\left(y(\mathbf{u})\right)/m^2 = \sigma^2 \left\langle C', C\right\rangle/m^2\end{aligned}$$

and (28) immediately follows from the orthogonality of characters. The usual matrix calculation rules can also be used to obtain $\operatorname{var}(\hat{\gamma})$, provided the transpose is replaced by the conjugate transpose: $\operatorname{var}(\hat{\gamma}) = Z^*(\sigma^2\mathbf{I})Z/m^2 = \sigma^2\mathbf{I}/m$. Note that $\operatorname{var}(x) = \operatorname{var}(\overline{x})$. Thus if $\operatorname{cov}(x, \overline{x}) = 0$, the variance of any real linear combination $ax + \overline{a}\,\overline{x}$ is $2|a|^2 \operatorname{var}(x)$. It is equal to $\operatorname{var}(x)$ if $|a| = 1/\sqrt{2}$.

Representation of the dual morphism

All levels in what follows will be embedded in the same cyclic group (M). The integer M must therefore be a common multiple of the orders of all the cyclic groups involved, that is of $n_1, \ldots, n_s, m_1, \ldots, m_r$ and $p_1, \ldots, p_q$. We let $\Phi = (\phi_{ij})$ be the matrix representing the morphism Φ (see Kobilinsky & Monod 1991). If $\mathbf{u} = (u_1, \ldots, u_r)^{\mathrm{t}}$, $\Phi\mathbf{u}$ is then a $s \times 1$ column vector with $\sum_j \phi_{ij} u_j$ as i^{th} coordinate. Thus

$$A_i(\Phi\mathbf{u}) = \frac{M}{n_i} \sum_j \phi_{ij} u_j = \sum_j \frac{\phi_{ij} m_j}{n_i} \frac{M}{m_j} u_j = \sum_j \phi^*_{ji} C_j(\mathbf{u})$$

that is

$$A_i \circ \Phi = \sum_j \phi^*_{ji} C_j \tag{31}$$

where $C_1, \ldots, C_r$ are the basic unit factors and

$$\phi^*_{ji} = \frac{\phi_{ij} m_j}{n_i} \tag{32}$$

Hence the image $A \circ \Phi$ of the character $A = a_1 A_1 + \cdots + a_s A_s$ associated with $\mathbf{a} = (a_1, \ldots, a_s)^{\mathrm{t}}$:

$$A \circ \Phi = \sum_i a_i \left(\phi^*_{1i} C_1 + \cdots + \phi^*_{ri} C_r\right) = \left(\sum_i \phi^*_{1i} a_i\right) C_1 + \cdots + \left(\sum_i \phi^*_{ri} a_i\right) C_r , \tag{33}$$

is the character of U associated with $(\sum_i \phi^*_{1i} a_i, \ldots, \sum_i \phi^*_{ri} a_i)$. With the representation of U^* and T^* provided by Proposition 2.1, the matrix of the dual of Φ is therefore $\Phi^* = (\phi^*_{ij})$. Note that by definition of the dual, we have $\Phi^*(A)(\mathbf{u}) = A\left(\Phi(\mathbf{u})\right)$ for all $\mathbf{u}$ in U and A in T^*. Using the matrix representation, this definition becomes

$$\forall \mathbf{u} \in U^*, \forall \mathbf{a} \in T^* \qquad [\Phi^*\mathbf{a}, \mathbf{u}] = [\mathbf{a}, \Phi\mathbf{u}] \tag{34}$$

or

$$\forall \mathbf{u} \in U^*, \forall \mathbf{a} \in T^* \qquad \eta^{[\Phi^* \mathbf{a}, \mathbf{u}]} = \eta^{[\mathbf{a}, \Phi \mathbf{u}]} \tag{35}$$

where η is the primitive M-root of unity given by (6).

The multiplicative form of (33)

$$A \circ \Phi = C_1^{\left(\sum_i \phi^*_{1i} a_i\right)} \cdots C_r^{\left(\sum_i \phi^*_{ri} a_i\right)} \tag{36}$$

can also be written

$$\eta^a \circ \Phi = \eta^{\Phi^* \mathbf{a}} \tag{37}$$

if we use as in (7) the notations $\eta^{\mathbf{a}}$ and $\eta^{\mathbf{c}}$ for the characters of T^* and U^* respectively associated with $\mathbf{a}$ and $\mathbf{c}$.

From (36) and the corresponding expression for $B \circ \Psi$, one can easily deduce which characters are confounded. A quicker way to get these results is to obtain first generators of $\operatorname{Ker} \Phi^*$ by the general technique given in El Mossadeq *et al.* (1985), then the whole subgroup $\operatorname{Ker} \Phi^*$ and finally for each character A the subset $A + \operatorname{Ker} \Phi^*$ of characters confounded with it. The corresponding coefficients $A(\mathbf{t}_0)$ and $B(\mathbf{v}_0)$ in (24) can be computed simultaneously.

As an example, consider the practically important case where $n_i = m_i$ and $A_i \circ \phi = C_i$ for $i = 1, \ldots, r$. That is the first r factors are defined on U by the projections on the r coordinates. Since $A_i \circ \phi = A_i(\mathbf{t}_0) A_i \circ \Phi$, we have, for each $i \leq r$, $A_i(\mathbf{t}_0) A_i \circ \Phi(0) = C_i(0)$, hence $A_i(\mathbf{t}_0) = 1$ and $A_i \circ \phi = A_i \circ \Phi = C_i$. For $i \leq r$ the coefficients ϕ^*_{ji} in (31) are therefore 0 if $i \neq j$ and 1 if $i = j$, and the matrix Φ^* has the identity $\mathbf{I}_r$ on its left.

For $i > r$, formula (31) written in multiplicative form gives

$$A_i \circ \phi = A_i(\mathbf{t}_0) \prod_{j=1}^{r} C_j^{\phi^*_{ji}}.$$

Up to multiplication by the coordinates of $\mathbf{t}_0$ in R_M, the factors $A_{r+1}, \ldots, A_s$ are thus defined as products of the basic factors.

It is easy to show from the above defining relations that the generators of $\operatorname{Ker} \Phi^*$ are the $s - r$ characters

$$F_i = \overline{A_i} \prod_{j=1}^{r} A_j^{\phi^*_{ji}} \ , i = r+1, \ldots, s, \tag{38}$$

$$\left(F_i = -A_i + \sum_{j=1}^{r} \phi^*_{ji} A_j \ , i = r+1, \ldots, s \quad \text{in additive form} \right) \tag{39}$$

(see Proposition (6.1) in Kobilinsky, Monod 1991) and that

$$F_i(\mathbf{t}_0) = \overline{A}_i(\mathbf{t}_0) \ . \tag{40}$$

So for each character $C = C_1^{f_1} \cdots C_r^{f_r}$

$$\gamma(C) = \sum_{f_{r+1}, \ldots, f_s} \overline{A}_{r+1}(\mathbf{t}_0)^{f_{r+1}} \cdots \overline{A}_s(\mathbf{t}_0)^{f_s} \, e\left(A_1^{f_1} \cdots A_r^{f_r} F_{r+1}^{f_{r+1}} \cdots F_s^{f_s} \right) \ . \tag{41}$$

if C is not confounded with a block character and

$$\gamma(C) = e(B) + \sum_{f_{r+1},\dots,f_s} \overline{A}_{r+1}(\mathbf{t}_0)^{f_{r+1}} \cdots \overline{A}_s(\mathbf{t}_0)^{f_s}\, e\left(A_1^{f_1} \cdots A_r^{f_r} F_{r+1}^{f_{r+1}} \cdots F_s^{f_s}\right) . \tag{42}$$

if C is confounded with the block character B and $B(\mathbf{v}_0) = 1$. The sum is over $(s-r)$-uplets $(f_{r+1}, \dots, f_s)$ in $(n_{r+1}) \times \cdots \times (n_s)$ and $(f_1, \dots, f_r)$ belongs to $(n_1) \times \cdots \times (n_r)$.

In order to lighten notations, the composition by ϕ is often omitted and the same notation is used for basic treatment and unit characters which coincide on U. Thus in the situation just considered, we say that the factors A_i for $i = r+1, \dots, s$ are defined from the basic unit factors $A_1, \dots, A_r$ by the rules

$$A_i = A_i(\mathbf{t}_0) A_1^{\phi_{1i}^*} \cdots A_r^{\phi_{ri}^*}$$

$$\left(\text{resp. } A_i = \frac{M}{n_i} t_{0i} + \phi_{1i}^* A_1 + \cdots + \phi_{ri}^* A_r \right).$$

Note that an effect $A \in T^*$ is confounded with C iff $A \circ \phi$ is proportional to C. The coefficient of proportionality is then $A(\mathbf{t}_0)$, and it is precisely the coefficient appearing in the expression (24) giving $\gamma(C)$. This result will often be used in the examples of Section 5.

5 Juxtaposition of GMG Designs: examples

The group morphism method gives no information at all for confounded contrasts and maximum information for unconfounded ones. This all black or all white procedure is often inappropriate. It is possible, however, in order to spread the loss of information more evenly, to apply this method separately to each of the subsets of a partition of the set U of units, confounding for instance different contrasts on the different subsets. This general principle of construction leads to the so-called lattice designs when the subsets are the distinct replicates. But it can be used successfully in many other circumstances. We give first a general framework for this kind of design, then illustrate by several examples.

5.1 General framework

The set T of treatments has the same structure as previously, but the set of experimental units U is now the disjoint union $U = U_1 \sqcup \cdots \sqcup U_K$ of K subsets $U_1, \dots, U_K$, each of which is identified with a product of cyclic groups. The set V associated with the blocks can similarly be the disjoint union of J products $V_1, \dots, V_J$ of cyclic groups: $V = V_1 \sqcup \cdots \sqcup V_J$.

The assignment of treatments and blocks is done on each U_k by the group morphism construction method. That is the treatment $\mathbf{t}$ and blocks $\mathbf{v}$ assigned to $\mathbf{u}$ in U_k are

$$\mathbf{t} = \phi_k(\mathbf{u}) = \mathbf{t}_k + \Phi_k \mathbf{u} \qquad \mathbf{v} = \psi_k(\mathbf{u}) = \mathbf{v}_k + \Psi_k \mathbf{u} . \tag{43}$$

Here Φ_k is a group morphism from U_k into T and $\mathbf{t}_k$ is a fixed element of T. Similarly Ψ_k is a group morphism from U_k into one of the subsets $V_1, \dots, V_J$, say V_j, and $\mathbf{v}_k$ is a fixed element of the same V_j. We denote by $\mathcal{J}$ the mapping giving for each $k = 1, \ldots, K$ the corresponding index j, so that

$$\operatorname{Im} \psi_k \subset V_{\mathcal{J}(k)} \ . \tag{44}$$

The mapping $\mathcal{J}$ may be the identity of K. In that case, any partition into blocks is nested within the partition U_1, ..., U_K. It may also be a non injective mapping. Then units in distinct subsets U_k and $U_{k'}$ can be assigned to the same block if $\mathcal{J}(k) = \mathcal{J}(k')$.

Note that the quantities indexed by k may be independent of k. For instance we may have $\Phi_k = \Phi$ for all k.

The model for the vector y_k of observations on the k^{th} subset U_k is

$$E(y_k) = \tau \circ \phi_k + \zeta_j \circ \psi_k \qquad \operatorname{var}(y_k) = \sigma^2 \mathbf{I} \tag{45}$$

where $j = \mathcal{J}(k)$ and ζ_j is the vector of block effects on V_j. Moreover the vectors y_1, ..., y_K are supposed independent, hence

$$\operatorname{Cov}(y_k, y_l) = \mathbf{0} \quad \text{for } k \neq l \ . \tag{46}$$

Decomposing τ and ζ_j on the orthogonal basis of characters gives

$$E(y_k) = \sum_A e(A) A \circ \phi_k + \sum_B e_j(B) B \circ \psi_k \tag{47}$$

or

$$E(y_k) = \sum_A A(\mathbf{t}_k) e(A) A \circ \Phi_k + \sum_B B(\mathbf{v}_k) e_j(B) B \circ \Psi_k \ . \tag{48}$$

The block effect $e_j(B)$ is defined by the following equality where $|V_j|$ denotes the number of elements of V_j:

$$e_j(B) = \langle \zeta_j, B \rangle / |V_j| \ . \tag{49}$$

Taking the scalar product with a character C of U_k in (48), we get:

$$E\Big(\langle y_k, C \rangle / |U_k|\Big) = \sum_{A : A \circ \Phi_k = C} A(\mathbf{t}_k) e(A) + \sum_{B : B \circ \Psi_k = C} B(\mathbf{v}_k) e_j(B) \ , \tag{50}$$

$$\begin{aligned} \operatorname{var}\left(\langle y_k, C \rangle / |U_k|\right) &= \frac{\sigma^2}{|U_k|} \ , \\ \operatorname{cov}\Big(\langle y_k, C \rangle / |U_k| \, , \, \langle y_{k'}, C' \rangle / |U_{k'}|\Big) &= 0 \qquad \text{if } k \neq k' \text{ or } C \neq C'. \end{aligned} \tag{51}$$

The subsets of canonical parameters $e(A)$, $e_j(B)$ associated in (50) with the different characters C of a given subset U_k are disjoints. This makes (50) a very convenient form of the model for practical use. Of course a given parameter can occur for different values of k, but it is generally easy to isolate disjoint equivalence classes of parameters and corresponding subsets of linear forms $\langle y_k, C \rangle / |U_k|$ such that the expectations in one subset involve only the parameters of the corresponding class. The estimation process can then be carried out separately for each subset and the corresponding information matrices are often easy to manipulate. The equivalence relation between parameters is generated by the sets of parameters appearing together in the expectation of one linear form. Two parameters are in the same equivalence class if there is a chain of parameters joining them such that two consecutive ones in the chain appear together in the same expectation.

The information matrix and least squares estimates (LSE) are invariant under any linear invertible transformation. Let indeed model (47) be written as

$$E(y) = X\theta \qquad \mathrm{var}(y) = \sigma^2 \mathbf{I} \tag{52}$$

where $y = (y_1^{\mathrm{t}}, \ldots, y_K^{\mathrm{t}})^{\mathrm{t}}$ is the vector of responses for the whole design, and let W be an invertible matrix (with coefficients in $\mathbf{C}$). Then

$$E(W^*y) = W^*X\theta \qquad \mathrm{var}(W^*y) = \sigma^2 W^*W \tag{53}$$

The information matrix and LSE computed from (53), $\Omega = X^*W(W^*W)^{-1}W^*X$ and $\hat{\theta} = \Omega^- X^*W(W^*W)^{-1}W^*y$, are clearly identical to those deduced from (52): $\Omega = X^*X$, $\hat{\theta} = \Omega^- X^*y$.

In particular, taking W such that the coordinates of W^*y are the linear forms on the left of (50), we see that the information matrix and LSE deduced from (50), with weight $|U_k|$ inversely proportional to the variances, are the same as those deduced directly from (47). In that case W^*W is diagonal with the inverses $1/|U_k|$ on the diagonal.

Let $x(A)$, $x_j(B)$ be the columns of X associated with $e(A)$, $e_j(B)$. The elements of the information matrix Ω are the scalar products between these columns. Using (50) we can express them as follows

$$\begin{aligned}
\langle x(A), x(A')\rangle &= \sum_{k\in K_1} |U_k| A(\mathbf{t}_k)\overline{A'}(\mathbf{t}_k)\ , \qquad & K_1 &= \{k / A\circ\Phi_k = A'\circ\Phi_k\} \\
\langle x(A), x_j(B)\rangle &= \sum_{k\in K_2} |U_k| A(\mathbf{t}_k)\overline{B}(\mathbf{v}_k)\ , \qquad & K_2 &= \{k / \mathcal{J}(k) = j,\ A\circ\Phi_k = B\circ\Psi_k\} \\
\langle x_j(B), x_j(B')\rangle &= \sum_{k\in K_3} |U_k| B(\mathbf{v}_k)\overline{B'}(\mathbf{v}_k)\ , \qquad & K_3 &= \{k / \mathcal{J}(k) = j,\ B\circ\Psi_k = B'\circ\Psi_k\} \\
\langle x_j(B), x_l(B')\rangle &= \ \ 0 & &\text{if } j \neq l
\end{aligned} \tag{54}$$

Note that the summations are over the indices k such that the two characters involved are confounded on U_k and which satisfy $\mathcal{J}(k) = j$ when one of the parameter is a block effect $e_j(B)$.

It is sometimes judicious to use different partitions of U, or even no partition, to compute different elements of the information matrix X^*X. For instance if the design contains r replicates of a subgroup T_0 of T and if U is identified with $T_0 \times (r)$, the assignment of treatments is defined by the morphism $\Gamma : (\mathbf{t}, i) \mapsto \mathbf{t}$ from $U = T_0 \times (r)$ into T. The columns $x(A)$ and $x(A')$ are the characters $A\circ\Gamma$ and $A'\circ\Gamma$. Their scalar product is $|U|$ if A and A' are confounded on T_0, 0 otherwise. In that case the partition of U can be useful for the computation of scalar products involving a block effect but is of no use for computation of scalar products involving only treatment effects.

5.2 Examples for factors at two levels

Example 1 . One replicate of a 2^5 in 8 blocks of size 4.

We can take $U = T = (2)^5$ and ϕ to be the identity. With the group morphism method, it is easy to show that, for any possible choice of Ψ, the subgroup $\operatorname{Im}\Psi^*$ of confounded treatment factorial effects includes some main effects or two-factor interactions.

If we assume however that interactions of three or more factors are negligible, we can estimate all main effects and interactions with a design built in the following manner. The five factors are denoted A, B, C, D, E and are identified with the basic multiplicative characters of $U = T$. The set U is split into two subsets, called macroblocks, according to the value of $ABCDE$. The macroblock 1, defined by $ABCDE = 1$ is then split into four blocks according to the values of ABC, BCD, while the macroblock 2, defined by $ABCDE = -1$ is divided into four blocks according to the values of ABD, ACD. In the resulting design, the main effects and some of the two-factor interactions, namely AB, AC, BD, CD, can be estimated with factor efficiency 1 without any assumption on the other interactions. The other two-factor interactions are estimated with efficiency 1/2 if it is assumed that interactions between three factors or more are zero. This is a consequence of a general result given in (7.1.2), but it is worth giving a direct detailed proof in that simple situation.

Consider the macroblock 1 defined by $ABCDE = 1$. We can identify its units with the elements of the group $U_1 = (2)^4$, with A, B, C, D as basic factors. The level of E is then defined by $E = ABCD$. and the block pseudofactors F and G by $F = ABC$, $G = BCD$. As indicated at the end of Section 4, these equalities are valid only on U_1. To be quite rigorous $A, \ldots, E$ and F, G should be replaced in them by $A \circ \phi_1, \ldots, E \circ \phi_1$ and $F \circ \psi_1$, $G \circ \psi_1$ where ϕ_1, ψ_1 are the affine mappings defining respectively the treatment and block on U_1.

On U_1, the four block characters $\mathbf{1}$, F, G, FG are confounded respectively with the unit characters $\mathbf{1}$, ABC, BCD, AD. The treatment characters confounded with a given unit character are the characters having the same literal expression and the one obtained by multiplication by $ABCDE$. For instance the treatment characters confounded with the unit character A are A and $BCDE$.

The CEFs on this macroblock are

- the sums of effects $e_1(\mathbf{1}) + e(\mathbf{1}) + e(ABCDE)$, $e_1(F) + e(ABC) + e(DE)$, $e_1(G) + e(BCD) + e(EF)$, $e_1(FG) + e(AD) + e(BCE)$ which include a block effect
- and those which do not, like $e(A) + e(BCDE)$, $e(AB) + e(CDE)$, etc ...

The effects indexed by 1 are defined by (49) from the 4×1 vector ζ_1 of block effects in this macroblock: $e_1(\mathbf{1}) = \langle \zeta_1, \mathbf{1} \rangle /4$, $e_1(F) = \langle \zeta_1, F \rangle /4$, $e_1(G) = \langle \zeta_1, G \rangle /4$, $e_1(FG) = \langle \zeta_1, FG \rangle /4$.

The above CEFs are estimated as in (28) by scalar products between the vector y_1 of responses on the macroblock and the corresponding unit character. For instance,

$e_1(G)+e(BCD)+e(EF)$ is estimated by $\langle y_1, BCD\rangle/16$, $e(A)+e(BCDE)$ by $\langle y_1, A\rangle/16$, etc ... Those estimators are uncorrelated of variance $\sigma^2/16$.

Similarly on macroblock 2 defined by $ABCDE = -1$, we have $F = ABD = -CE$, $G = ACD = -BE$, $FG = BC = -ADE$. The CEFs are

- $e_2(\mathbf{1}) + e(\mathbf{1}) - e(ABCDE)$, $e_2(F) + e(ABD) - e(CE)$, $e_2(G) + e(ACD) - e(BE)$, $e_2(FG) + e(BC) - e(ADE)$ which include a block effect,
- and the differences between confounded treatment effects like $e(A) - e(BCDE)$, $e(AB) - e(CDE)$, etc ...

The corresponding estimators $\langle y_2, \mathbf{1}\rangle/16$, $\langle \mathbf{y}_2, ABD\rangle/16$, etc ... are uncorrelated of variance $\sigma^2/16$ and have no correlation with those of the first macroblock.

Now from a pair like $e(A)+e(BCDE)$, $e(A)-e(BCDE)$ in which there is no block effect, we can deduce estimates of variance $\sigma^2/32$ of $e(A)$ and $e(BCDE)$. All factorial effects which are never confounded with a block effect can therefore be estimated without any increase of the variance due to the blocks, that is with efficiency 1.

From the two contrasts $\langle y_1, ABC\rangle/16$, $\langle \mathbf{y}_2, ABC\rangle/16$ bearing information on the three parameters $e_1(F)$, $e(ABC)$, $e(DE)$, no separate estimate of these parameters can be obtained unless $e(ABC)$ is assumed to be zero. In that case the two contrasts are least squares estimates of $e_1(F) + e(DE)$ and $-e(DE)$ respectively. The second one estimates the interaction DE with a variance $\sigma^2/16$. This is twice the variance $\sigma^2/32$ which would be obtained in a design letting this interaction be totally unconfounded, and the corresponding efficiency is therefore 1/2.

Example 2 . Two replicates of a 2^5 in blocks of size 4.

If the 2^5 treatments can be experimented in a second replicate, it is possible, by a suitable blocking, to estimate all factorial effects except the five factor interaction $ABCDE$. The blocking process is similar to the one used for the first replicate. The division into two macroblocks, numbered 3 and 4, is equally made according to the value of $ABCDE$. To make the subsequent division into blocks, several solutions are possible, for instance

- **Solution 1.** Lattice type design.

  One uses the same pseudofactors as for the first replicate, but it is macroblock 4, defined by $ABCDE = -1$ which is split according to the values of ABC and BCD, while macroblock 3 defined by $ABCDE = 1$ is split according to the values of ABD and ACD. The resulting design can also be described as being made up of a first replicate (macroblocks 1 and 4) on which blocks are defined by the values of the three pseudofactors $ABCDE$, ABC, BCD, and of a second replicate (macroblocks 2 and 3) on which blocks are defined by the values of $ABCDE$, ABD, ACD. The factor efficiencies are 0 for $ABCDE$, 1/2 for ABC, DE, BCD, AE, AD, BCE, ABD, CE, ACD, BE, BC, ADE which are confounded with blocks in one of the replicates, and 1 for the remaining eighteen factorial effects.

- **Solution 2.** Balancing the loss of information over macroblocks.

  The unit pseudofactors used to split macroblocks 3 and 4 into blocks are chosen so that the subgroups of confounded unit characters have only **1** in common. For instance, AB, CD are used for macroblock 3 and A, C for macroblock 4. The confounded subgroups of unit and treatment characters are given in Table 1. The only treatment characters confounded with the blocks on more than one macroblock are **1**, $ABCDE$, and they are in fact confounded in every macroblock. The efficiency is 1 for unconfounded treatment effects, 2/3 for treatment effects confounded in one of the macroblocks, and 0 for $ABCDE$. Let us prove for instance that the efficiency is 2/3 for ABC and DE, confounded with F in the first macroblock. We have

macroblock	subgroup of confounded unit characters	subgroup of confounded treatment characters
1 $(ABCDE = 1)$	$\{\mathbf{1}, ABC, BCD, AD\}$	$\left\{\begin{matrix}\mathbf{1}, ABC, BCD, AD,\\ ABCDE, DE, AE, BCE\end{matrix}\right\}$
2 $(ABCDE = -1)$	$\{\mathbf{1}, ABD, ACD, BC\}$	$\left\{\begin{matrix}\mathbf{1}, ABD, ACD, BC,\\ ABCDE, CE, BE, ADE\end{matrix}\right\}$
3 $(ABCDE = 1)$	$\{\mathbf{1}, AB, CD, ABCD\}$	$\left\{\begin{matrix}\mathbf{1}, AB, CD, ABCD,\\ ABCDE, CDE, ABE, E\end{matrix}\right\}$
4 $(ABCDE = -1)$	$\{\mathbf{1}, A, C, AC\}$	$\left\{\begin{matrix}\mathbf{1}, A, C, AC,\\ ABCDE, BCDE, ABDE, BDE\end{matrix}\right\}$

Table 1: *Effects confounded with blocks in each macroblock of a two replicate* 2^5

$$\begin{array}{rclllll} E(\langle y_1, ABC\rangle/16) &=& e_1(F) &+& e(ABC) + e(DE) &=& \alpha \\ E(\langle y_2, ABC\rangle/16) &=& && e(ABC) - e(DE) &=& \beta \\ E(\langle y_3, ABC\rangle/16) &=& && e(ABC) + e(DE) &=& \gamma \\ E(\langle y_4, ABC\rangle/16) &=& && e(ABC) - e(DE) &=& \beta \end{array}$$

and among all contrasts of the form $\langle y_k, C\rangle/16$, these contrasts are the only ones bearing information on these parameters. Clearly the least squares estimates of β and γ and their variances are

$$\hat\beta = \frac{(\langle y_2, ABC\rangle/16) + (\langle y_4, ABC\rangle/16)}{2} \qquad \text{var}(\hat\beta) = \frac{\sigma^2}{32}$$

$$\hat\gamma = \langle y_3, ABC\rangle/16 \qquad \text{var}(\hat\gamma) = \frac{\sigma^2}{16}.$$

Hence the least squares estimates of $e(ABC)$, $e(DE)$ and corresponding variances are

$$\hat e(ABC) = \frac{\hat\gamma + \hat\beta}{2} \qquad \text{var}(\hat e(ABC)) = \frac{3}{2}\frac{\sigma^2}{64}$$

$$\hat e(DE) = \frac{\hat\gamma - \hat\beta}{2} \qquad \text{var}(\hat e(DE)) = \frac{3}{2}\frac{\sigma^2}{64}$$

Comparing these variances with the variance $\sigma^2/64$ of an unconfounded factorial effect, we find the announced efficiency of 2/3. It is also interesting to note that if the four factor interactions $BCDE$ and $ABDE$ are assumed to be zero, the main effects A and C are estimated with efficiency 3/4 instead of 2/3. These factor efficiencies can also be deduced from the general results of Section 7.1.2.

- **Solution 3.** An intermediate design.

  To avoid losing efficiency on some main effects, we can use AB, CD to split macroblock 3 as before, but again ABC and BCD to split macroblock 4. The efficiency is then 1/2 for ABC, DE, BCD, AE AD, BCE as in the lattice type design, but increases to 2/3 for the factorial effects confounded in macroblock 2 and 3. This design thus gives a better balance for estimating interactions than the lattice type, while keeping full efficiency on the main effects A, B, C, D.

5.3 Block design for factors at two and three levels

Example 3 . One replicate of a $3 \times 3 \times 3 \times 2$ in blocks of size 6.

Yates (1937) gives two-replicate designs for 4 factors A,B,C,D with 3, 3, 3, 2 levels respectively, in blocks of size 6. A similar one-replicate design is given by Kempthorne (1952) and further studied by Winer (1962). However, the results on efficiency given by the last two authors are inaccurate because they fail to take into account the destruction of orthogonality between treatment effects caused by the adjustment for blocks. So, it is again worth studying one-replicate designs of this kind. We do this for a single replicate coming from Yates' designs, because in contrast to Kempthorne's, the Yates' replicates can be obtained as disjoint unions of two designs built by the group morphism method.

The replicate studied here is the first replicate of the Z-design given in Yates' Table 69. It is made up of two disjoint subsets U_1 and U_2 of 3^3 units, which are identified with $(3)^3$ with A, B, C as basic factors (this means that the levels of the factors A, B, C are precisely given by the three coordinates). The fourth two-level treatment factor D and the two three-level block pseudofactors denoted by P, Q are defined

- on U_1 by : $D = \mathbf{1}$, $P = AB^2$, $Q = AC^2$
- on U_2 by : $D = -\mathbf{1}$, $P = \mathrm{j}AB^2$, $Q = \mathrm{j}^2AC^2$

where $\mathrm{j} = exp(2\pi \mathrm{i}/3)$ is a primitive cubic root of unity in $\mathbf{C}$.

We have for any couple $(p, q) \in (3)^2$

$$P^pQ^q = A^{p+q}B^{2p}C^{2q}, \quad A^{p+q}B^{2p}C^{2q}D = A^{p+q}B^{2p}C^{2q} \quad \text{on } U_1$$

$$P^pQ^q = j^{p+2q}A^{p+q}B^{2p}C^{2q}, \quad A^{p+q}B^{2p}C^{2q}D = -A^{p+q}B^{2p}C^{2q} \quad \text{on } U_2$$

Therefore the CEFs including the block effect P^pQ^q are

$$\begin{array}{lllllll} \gamma_1 & = & e(A^{p+q}B^{2p}C^{2q}) & + & e(A^{p+q}B^{2p}C^{2q}D) & + & e(P^pQ^q) & \text{on } U_1 \\ \gamma_2 & = & e(A^{p+q}B^{2p}C^{2q}) & - & e(A^{p+q}B^{2p}C^{2q}D) & + & \mathrm{j}^{p+2q}e(P^pQ^q) & \text{on } U_2. \end{array} \quad (55)$$

If $\mathrm{j}^{p+2q} = 1$, that is $p = q$, the two effects $A^{p+q}B^{2p}C^{2q} = A^{2q}B^{2q}C^{2q}$ and P^pQ^q are confounded. Thus ABC and $A^2B^2C^2$ are not estimable. On the other hand, $A^{2q}B^{2q}C^{2q}D$ is estimated in that case by $(\hat{\gamma}_1 - \hat{\gamma}_2)/2$ with efficiency 1. In particular if $p = q = 0$, this shows that D can be estimated with efficiency 1.

If $p \neq q$, none of the three factorial effects in (55) is estimable unless one of them is assumed to be zero. Note that in that case, $A^{p+q}B^{2p}C^{2q}$ is one of the six two-factor interactions AB^2, A^2B, AC^2, A^2C, BC^2, B^2C. If we assume that the three-factor interaction $A^{p+q}B^{2p}C^{2q}D$ is zero, we have

$$e(A^{p+q}B^{2p}C^{2q}) = \frac{\mathrm{j}^{p+2q}\gamma_1 - \gamma_2}{\mathrm{j}^{p+2q} - 1} \qquad e(P^pQ^q) = \frac{-\gamma_1 + \gamma_2}{\mathrm{j}^{p+2q} - 1}$$

To get the corresponding estimates, we replace γ_1 and γ_2 by their estimates. Since $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are uncorrelated of variance $\sigma^2/27$, the variance of $\hat{e}(A^{p+q}B^{2p}C^{2q})$ is

$$\operatorname{var}\left(\hat{e}(A^{p+q}B^{2p}C^{2q})\right) = \frac{\left(\left|j^{p+2q}\right|^2 \frac{\sigma^2}{27} + \frac{\sigma^2}{27}\right)}{\left|j^{p+2q} - 1\right|^2} = \frac{2}{3}\frac{\sigma^2}{27}$$

Comparing it with the variance $\frac{1}{2}\frac{\sigma^2}{27}$ for unconfounded contrasts, we get $3/4$ for the corresponding efficiency. The efficiency for the three-factor interaction $A^{p+q}B^{2p}C^{2q}D$ is similarly found to be $1/4$ if the two-factor interaction $A^{p+q}B^{2p}C^{2q}$ is zero. But this last result has no interest from the practical point of view.

It is easy to check that the other parameters, which are confounded with a block effect neither on U_1 nor on U_2, are estimated with efficiency 1 without making any hypothesis on the form of treatment effects. Table 2 sums up these results.

AB^2	$(3/4, 0)$	,	AB^2D	$(1/4, 0)$
A^2B	$-$	,	A^2BD	$-$
AC^2	$-$	,	AC^2D	$-$
A^2C	$-$	,	A^2CD	$-$
B^2C	$-$	,	B^2CD	$-$
BC^2	$-$	,	BC^2D	$-$
$A^2B^2C^2$	$(0, 0)$	,	$A^2B^2C^2D$	$(1, 1)$
ABC	$-$	,	$ABCD$	$-$
$\mathbf{1}$	$(0, 0)$	,	D	$(1, 1)$

Table 2: *Efficiencies for confounded treatment effects in Example 3*

The 2 efficiencies in brackets correspond to the absence or presence of the second parameter of the line in the model. The efficiencies are 1 for all the parameters not listed in this table.

In this example conjugate contrasts such as $e(AB^2)$, $e(A^2B)$ are estimated from two uncorrelated sets of linear forms of the responses $\{\langle y_1, AB^2\rangle, \langle y_2, AB^2\rangle\}$ and $\{\langle y_1, A^2B\rangle,$

$\langle y_2, A^2B\rangle\}$. So their estimates are uncorrelated and the variance of estimation of any real linear combination $a\, e(AB^2) + \overline{a}\, e(A^2B)$ with $|a| = 1/\sqrt{2}$ is equal to the variance of estimation of $e(AB^2)$. Therefore any real linear combination of $e(AB^2)$ and $e(A^2B)$ is estimated with the same efficiency as $e(AB^2)$, that is $3/4$ if $e(AB^2D) = e(A^2BD) = 0$.

For a similar reason the estimates of the four canonical contrasts $e(AB^2)$, $e(A^2B)$, $e(AB)$, $e(A^2B^2)$ spanning the interaction between A and B are uncorrelated. The variance of estimation of any real linear combination $a\, e(AB^2) + \overline{a}\, e(A^2B) + b\, e(AB) + \overline{b}\, e(A^2B^2)$ is $2(|a|^2 4/3 + |b|^2)\sigma^2/54$ if block effects are taken into account and $2(|a|^2 + |b|^2)\sigma^2/54$ if they are not. The corresponding efficiency $(|a|^2 + |b|^2)/(|a|^2 4/3 + |b|^2)$ varies between $3/4$ and 1. These two extreme efficiencies are called the *principal efficiencies* for the estimation of the interaction $A \times B$. Principal efficiencies are defined more generally and precisely in Section 6.

Example 4 . Two replicates of a $3 \times 3 \times 3 \times 2$ in blocks of size 6.

- **Solution 1** Yates' design.

  If one can afford a second replicate, it can be constructed with Yates' method by exchanging levels 1 and -1 of factor D in the first replicate. The resulting design includes four disjoint subsets U_1, U_2, U_3, U_4 which are identified with $(3)^3$. The factors D, P, Q and the replicate factor denoted by R are defined from the basic factors A, B, C by:

$$\begin{array}{lllll}
D = \;\;1 & P = \;AB^2 & Q = \;\;AC^2 & R = \;\;1 & \text{on } U_1 \\
D = -1 & P = \mathrm{j}AB^2 & Q = \mathrm{j}^2AC^2 & R = \;\;1 & \text{on } U_2 \\
D = -1 & P = \;AB^2 & Q = \;\;AC^2 & R = -1 & \text{on } U_3 \\
D = \;\;1 & P = \mathrm{j}AB^2 & Q = \mathrm{j}^2AC^2 & R = -1 & \text{on } U_4
\end{array}$$

  This design can be constructed more simply as the disjoint union of two subsets $U_{13} = U_1 \sqcup U_3$ and $U_{24} = U_2 \sqcup U_4$ which are identified with $(3)^3 \times (2)$. The block factors P, Q, R are then defined from the four basic factors A, B, C, D by

$$\begin{array}{llll}
P = \;AB^2 & Q = \;\;AC^2 & R = \;\;D & \text{on } U_{13} \\
P = \mathrm{j}AB^2 & Q = \mathrm{j}^2AC^2 & R = -D & \text{on } U_{24}
\end{array}$$

  The efficiencies for the confounded effects generated by AB^2, AC^2, D are then those given by the first number in brackets in Table 2. Note that it is not necessary in that case to make any hypothesis on treatment effects. Unconfounded effects are estimated with efficiency 1.

- **Solution 2**

  Instead of confounding the same factorial effects in the two subsets U_{13} and U_{24} as in Yates' design, we can define P, Q, R by

$$\begin{array}{llll}
P = \;ABC & Q = AB^2 & R = \;\;D & \text{on } U_{13} \\
P = \mathrm{j}ABC & Q = \;\;A & R = -D & \text{on } U_{24}
\end{array}$$

Again the efficiency is 1 for any factorial effect which is confounded neither on U_{13} nor on U_{24}. The other effects can be dissociated into those which are confounded on both subsets and those which are confounded only on one of them.

Any factorial effect $(ABC)^p D^r$ where $(p, r) \in (3) \times (2)$ is confounded in both subsets and the corresponding CEFs are

$$\begin{array}{rcl} \gamma_{13} & = & e(A^pB^pC^pD^r) \ + \ e(P^pR^r) \\ \gamma_{24} & = & e(A^pB^pC^pD^r) \ + \ \mathrm{j}^p(-1)^r e(P^pR^r) \quad . \end{array}$$

Hence $e(A^pB^pC^pD^r)$ is estimated by $(\hat{\gamma}_{24} - \mathrm{j}^p(-1)^r\hat{\gamma}_{13})/(1 - \mathrm{j}^p(-1)^r)$ when $(p, r) \neq (0, 0)$, with variance $(2\sigma^2/54)/|1 - \mathrm{j}^p(-1)^r|^2$ and efficiency $|1 - \mathrm{j}^p(-1)^r|^2/4$ (see Table 3).

The factorial treatment effects confounded only on one subset are those which are confounded either on U_{13} or on U_{24} with a block effect $P^pQ^qR^r$ where $(p, q, r) \in (3) \times (3) \times (2)$ and $q \neq 0$. The corresponding CEFs are

$$\begin{array}{lcl} \gamma_{13}\left((ABC)^p(AB^2)^qD^r\right) & = & e\left((ABC)^p(AB^2)^qD^r\right) + e(P^pQ^qR^r) \\ \gamma_{13}\left((ABC)^pA^qD^r\right) & = & e\left((ABC)^pA^qD^r\right) \\ \gamma_{24}\left((ABC)^p(AB^2)^qD^r\right) & = & e\left((ABC)^p(AB^2)^qD^r\right) \\ \gamma_{24}\left((ABC)^pA^qD^r\right) & = & e\left((ABC)^pA^qD^r\right) + (-1)^r\mathrm{j}^p e(P^pQ^qR^r) \end{array}$$

Let X be the corresponding 4×3 matrix of coefficients. Then if $a = (-1)^r\mathrm{j}^p$

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & a \end{bmatrix}, \quad X^*X = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & a \\ 1 & \overline{a} & 2 \end{bmatrix}, \quad (X^*X)^{-1} = \frac{1}{4}\begin{bmatrix} 3 & \overline{a} & -2 \\ a & 3 & -2a \\ -2 & -2\overline{a} & 4 \end{bmatrix}.$$

The variance of estimation is then $\frac{3}{4}\frac{\sigma^2}{54}$ and the efficiency 2/3 for any of these factorial treatment effects which are also listed in Table 3. Thus if we accept to reduce the efficiency from 1 to 2/3 for the main effect of A, all interactions of 2 and 3 factors (including ABC and $A^2B^2C^2$ which could not be estimated in Yates' design) are estimated intra-block with a minimum efficiency of 2/3.

Example 5 . DSIGN method with a non **Z**-linear key matrix.

Patterson (1976) uses the DSIGN method with a key matrix which does not define a group morphism to generate a $3 \times 2 \times 2$ in two blocks of six plots. The set U of units is identified with $(2) \times (3) \times (2)$. The unit $\mathbf{u} = (u_1, u_2, u_3)$ is allocated to the block u_1 and receives the treatment $\mathbf{t} = (t_1, t_2, t_3)$ defined by

$$t_1 = u_2, \quad t_2 = u_1 + u_2 + u_3 \pmod 2, \quad t_3 = u_3$$

The above rule is somewhat ambiguous, since $u_2 = 0$ and $u_2 = 3$ which are equal in (3) are not congruent modulo 2. We will therefore suppose that u_2 takes the values 0, 1, 2 only. The treatment is then well defined, but the above rules, which send the element $\mathbf{u} = (0, 1, 0)$ of period 3 on the element $\mathbf{t} = (1, 1, 0)$ of period 6, do not define a morphism.

	p	0	1	2	0	1	2
	r	0	0	0	1	1	1
q							
0		$\mathbf{1}$	ABC	$A^2B^2C^2$	D	$ABCD$	$A^2B^2C^2D$
efficiency		0	3/4		1	1/4	
1		AB^2	A^2C	BC^2	AB^2D	A^2CD	BC^2D
2		A^2B	B^2C	AC^2	A^2BD	B^2CD	AC^2D
1		A	A^2BC	B^2C^2	AD	A^2BCD	B^2C^2D
2		A^2	BC	AB^2C^2	A^2D	BCD	AB^2C^2D
efficiency		2/3					

Table 3: *Example 4, solution 2. Effects $(ABC)^p(AB^2)^qD^r$ or $(ABC)^pA^qD^r$ partially confounded with blocks and their factor efficiencies*

Let A, B, C be the factors respectively associated with t_1, t_2, t_3 and P the factor block. For $k = 0, 1, 2$ let U_k be the subset of U defined by $u_2 = k$. This subset can be identified with the group $(2) \times (2)$ with P, C as basic factors. The above rules then give

$$A = \mathrm{j}^k \mathbf{1}, \quad B = (-1)^k PC \quad \text{on } U_k$$

On U_k, the estimable linear combination of confounded parameters including a block parameter are

$$\gamma_k(\mathbf{1}) = e(\mathbf{1}_B) + e(\mathbf{1}_T) + \mathrm{j}^k e(A) + \mathrm{j}^{2k} e(A^2) \tag{56}$$

$$\gamma_k(P) = e(P) + (-1)^k e(BC) + (-\mathrm{j})^k e(ABC) + (-\mathrm{j}^2)^k e(A^2BC) \tag{57}$$

From the three contrasts $\hat{\gamma}_k(\mathbf{1})$, the conjugated parameters $e(A)$ and $e(A^2)$ can be estimated with efficiency 1. From the three contrasts $\hat{\gamma}_k(P)$, $e(BC)$ can be estimated if $e(ABC)$ and $e(A^2BC)$ are zero, that is if there is no three-factor interaction. In that case, the matrix X of coefficients of $e(P)$ and $e(BC)$ and the corresponding matrices X^*X and $(X^*X)^{-1}$ are

$$X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad X^*X = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad (X^*X)^{-1} = \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

The variance of $\hat{e}(BC)$ is $\frac{3}{8}\frac{\sigma^2}{4}$ and the efficiency $\frac{1/3}{3/8} = 8/9$. The other unconfounded parameters are estimated with efficiency 1.

5.4 Semi-regular fractional designs

P.W.M. John (1962) developed the concept of a three quarter fractional factorial for 2-level factors. Let $T = (2)^s$ be the group of treatments, $\mathbf{t} + T_0$ a regular fraction, G a subgroup of index 4 in T_0. A three-quarter fractional built on G, T_0 is made up of three out of the four cosets of G in $\mathbf{t} + T_0$. The approach of John can be used more generally with arbitrary groups and number K of cosets. The non-orthogonal fractions of the $2^m 3^n$ given by Connor & Young (1961) are obtained in exactly the same manner. We give other examples below. General rules to compute factor efficiencies when there is just one missing coset ($K = (T_0 : G) - 1$) or when there there are only $K = 2$ cosets are given in Sections 7.1.1 and 7.2.

Example 6 . 2/3 of a 3^4.

Let A, B, C, D be the 4 three-level factors. The 1/3 fraction defined by $ABCD = 1$ is of resolution 4. It does not allow for the estimation of two-factor interactions and can estimate main effects only if three-factor interactions are assumed to be zero. However, it is not necessary to experiment the whole complete factorial design (3^4 treatments) in order to get resolution 5. It is enough to experiment a 2/3 fraction, for instance the one including the two 1/3 fractions defined by $ABCD = 1$, $ABCD = \mathrm{j}$. Main effects are then estimable with efficiency 3/4 if four-factor interactions are assumed to be zero. The same is true for one half of the two-factor interaction contrasts. The other half is estimated with efficiency 1 if the three-factor interactions are assumed to be zero.

The proof follows the same lines as in the previous examples. Each fraction is identified with $(3)^3$ with A, B, C as basic factors. The CEFs associated with a unit contrast $A^a B^b C^c$ are

$$\gamma_1(A^a B^b C^c) = \sum_{i=0}^{2} e\left(A^a B^b C^c (ABCD)^i\right)$$

$$\gamma_2(A^a B^b C^c) = \sum_{i=0}^{2} \mathrm{j}^i\, e\left(A^a B^b C^c (ABCD)^i\right)$$

In particular those associated with A are

$$\begin{aligned} \gamma_1(A) &= e(A) + e(A^2BCD) + e(B^2C^2D^2) \\ \gamma_2(A) &= e(A) + \mathrm{j}e(A^2BCD) + \mathrm{j}^2 e(B^2C^2D^2) \end{aligned}$$

It follows that if $e(A^2BCD) = 0$, $e(A)$ is estimated with variance $(2/3)(\sigma^2/27)$ and efficiency $(1/2)/(2/3) = 3/4$. The same is true for the conjugated contrast $e(A^2)$ and for any other main effect by symmetry.

Similarly

$$\begin{aligned} \gamma_1(AB) &= e(AB) + e(A^2B^2CD) + e(C^2D^2) \\ \gamma_2(AB) &= e(AB) + \mathrm{j}e(A^2B^2CD) + \mathrm{j}^2 e(C^2D^2) \end{aligned}$$

hence $e(AB)$ and therefore the conjugated contrast $e(A^2B^2)$ are estimated with efficiency 3/4 when $e(A^2B^2CD) = e(ABC^2D^2) = 0$. The same is of course true for all interaction contrasts of the same form like AC, A^2C^2, BC, B^2C^2, etc ...

Finally

$$\begin{aligned} \gamma_1(AB^2) &= e(AB^2) + e(A^2CD) + e(BC^2D^2) \\ \gamma_2(AB^2) &= e(AB^2) + \mathrm{j}e(A^2CD) + \mathrm{j}^2 e(BC^2D^2) \end{aligned}$$

hence $e(AB^2)$ and the conjugated contrast $e(A^2B)$ are estimated with efficiency 1 when the three-factor interactions are zero. The same is true for all interaction contrasts of the same form like AC^2, A^2C, etc ...

Example 7 . 2/6 of a $2^2 \times 3^4$.

We let A, B be the two-level factors, C, D, E, F the three-level ones and select two out of the six 1/6 fractions respectively defined by $\{AB = 1, CDEF = 1\}$, $\{AB = -1, CDEF = 1\}$, $\{AB = 1, CDEF = \mathrm{j}\}$, $\{AB = -1, CDEF = \mathrm{j}\}$, $\{AB = 1, CDEF = \mathrm{j}^2\}$ and $\{AB = -1, CDEF = \mathrm{j}^2\}$.

Note that if the two selected fractions have the same value of $CDEF$, the design is a classical 1/3 regular fraction with defining character $CDEF$ having a resolution of 4 only. If the two selected fractions have the same value of AB, the design is included in a resolution 2 fraction with defining character AB and does not even allow the separate estimation of A and B.

Let us examine the case of two fractions for which neither AB nor $CDEF$ have the same value, for instance those defined by $\{AB = 1, CDEF = 1\}$ and $\{AB = -1, CDEF = \mathrm{j}\}$. We can take A, C, D, E as basic factors on each fraction. The estimable linear combinations of confounded parameters are

$$\gamma_1(A^aC^cD^dE^e) = \sum_{i=0}^{1}\sum_{k=0}^{2} e\left(A^aC^cD^dE^e(AB)^i(CDEF)^k\right)$$

$$\gamma_2(A^aC^cD^dE^e) = \sum_{i=0}^{1}\sum_{k=0}^{2}(-1)^i\mathrm{j}^k e\left(A^aC^cD^dE^e(AB)^i(CDEF)^k\right)$$

For each pair of conjugated parameters, it is enough to examine only one of them. Moreover, thanks to the symmetries, we can restrict our study of main effects and two-factor interactions to A, C, AB, AC, CD, CD^2. The corresponding subsets of confounded parameters are given in Table 4. Table 5 gives on the two first lines of each cell

$\Rightarrow \mathbf{1}$ $CDEF$ $C^2D^2E^2F^2$	$\Rightarrow AB$ $ABCDEF$ $ABC^2D^2E^2F^2$	$\Rightarrow A$ $ACDEF$ $AC^2D^2E^2F^2$	$\Rightarrow B$ $BCDEF$ $BC^2D^2E^2F^2$
$\Rightarrow C$ C^2DEF $D^2E^2F^2$	ABC ABC^2DEF $ABD^2E^2F^2$	$\Rightarrow AC$ AC^2DEF $AD^2E^2F^2$	$\Rightarrow BC$ BC^2DEF $BD^2E^2F^2$
$\Rightarrow CD$ C^2D^2EF $\Rightarrow E^2F^2$	$ABCD$ ABC^2D^2EF ABE^2F^2	$\Rightarrow CD^2$ C^2EF DE^2F^2	$ABCD^2$ ABC^2EF $ABDE^2F^2$

Table 4: *Example 7. Confounded parameters on the* 1/6 *fraction*
(main effects and two-factor interactions are marked with an arrow)

the corresponding CEFs when interactions of four factors or more are supposed to be zero. Under that hypothesis, $\mathbf{1}$, A, AB, AC can be estimated with efficiency 1, CD with efficiency 3/4, but C and CD^2 cannot be estimated. These last two effects can be estimated with efficiency 1 if the three-factor interactions are also assumed to be zero. Under that latter hypothesis, all main effects and two-factor interactions are therefore

$e(\mathbf{1})\ +\ e(AB)$ $e(\mathbf{1})\ -\ e(AB)$ $[e(\mathbf{1})\ +\ e(AB)]$	$e(A)\ +\ e(B)$ $e(A)\ -\ e(B)$ $[e(A)\ +\ e(B)]$
$e(C)\ +\ e(ABC)\ +\ e(D^2E^2F^2)$ $e(C)\ -\ e(ABC)\ +\ \mathrm{j}^2e(D^2E^2F^2)$ $[e(C)\ +\ e(ABC)\ +\ \mathrm{j}e(D^2E^2F^2)]$	$e(AC)\ +\ e(BC)$ $e(AC)\ -\ e(BC)$ $[e(AC)\ +\ e(BC)]$
$e(CD)\ +\ e(E^2F^2)$ $e(CD)\ +\ \mathrm{j}^2e(E^2F^2)$ $[e(CD)\ +\ \mathrm{j}e(E^2F^2)]$	$e(CD^2)\ +\ e(C^2EF)\ +\ e(DE^2F^2)$ $e(CD^2)\ +\ \mathrm{j}e(C^2EF)\ +\ \mathrm{j}^2e(DE^2F^2)$ $[e(CD^2)\ +\ \mathrm{j}^2e(C^2EF)\ +\ \mathrm{j}e(DE^2F^2)]$

Table 5: *Example 7. Canonical estimable functions*

estimable. The corresponding efficiencies are 3/4 for interactions like CD, C^2D^2 and 1 otherwise. Thus, the design is of resolution 5.

In their list of resolution 5 fractions of the 2^m3^n, Connor & Young (1961) propose for the 2^23^4 the 1/2 fraction made up of the three 1/6 fractions defined by $\{AB = 1, CDEF = 1\}$, $\{AB = -1, CDEF = \mathrm{j}\}$, $\{AB = 1, CDEF = \mathrm{j}^2\}$. The CEFs for the 54 units of the supplementary fraction $\{AB = 1, CDEF = \mathrm{j}^2\}$ are given in brackets in Table 5. It is easy to check that all main effects and two-factor interactions can be estimated even when three-factor interactions are not assumed to be zero. The corresponding efficiencies are 8/9 for contrasts like AB, A, AC, 4/5 for contrasts like C and 1 for contrasts like CD, CD^2. The three-factor interactions in Table 5 can also be estimated, but the four confounded interactions ACD, BCD, AE^2F^2, BE^2F^2 cannot be estimated from the three corresponding CEFs. The Connor and Young fraction has therefore resolution 6 (but not 7). However in most circumstances, the resolution 5 achieved by the 1/3 fraction of the example will be enough for practical purposes.

6 Real reparametrisation of complex linear models

As outlined in Section 3, a possible way to study a *complex* model like (16) is to recombine each pair of conjugate parameters $e(A)$, $e(\overline{A})$ in θ into real parameters $r(A)$, $r(\overline{A})$ through a unitary transformation, for instance

$$\begin{bmatrix} r(A) \\ r(\overline{A}) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -\mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} e(A) \\ e(\overline{A}) \end{bmatrix}, \tag{58}$$

or equivalently since the 2×2 matrix in (58) is unitary

$$\begin{bmatrix} e(A) \\ e(\overline{A}) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & \mathrm{i}/\sqrt{2} \\ 1/\sqrt{2} & -\mathrm{i}/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} r(A) \\ r(\overline{A}) \end{bmatrix}. \tag{59}$$

Consider a fixed factorial effect and suppose θ is partitioned into the vector θ_1 of parameters pertaining to this effect and the vector θ_0 of other parameters: $\theta^{\mathrm{t}} = (\theta_0^{\mathrm{t}}, \theta_1^{\mathrm{t}})$.

Let X be partitioned accordingly: $X = (X_0, X_1)$. The model (16) can be written:

$$E(y) = X_0\theta_0 + X_1\theta_1 \tag{60}$$

or, as function of the real parameters defined by (58):

$$E(y) = X_0 H_0 \zeta_0 + X_1 H_1 \zeta_1 \tag{61}$$

where H_0 and H_1 are the unitary matrices deduced from (59) (note that X_0H_0 and X_1H_1 are real) and ζ_0, ζ_1 are the vectors of real parameters corresponding to θ_0 and θ_1 respectively.

To get the least squares estimate of ζ_1 or θ_1, we introduce the operator Q_0 of orthogonal projection onto the orthogonal $(\operatorname{Im} X_0H_0)^\perp$ of the subspace generated by the columns of X_0H_0: $Q_0 = \mathbf{I} - X_0H_0(H_0^*X_0^*X_0H_0)^- H_0^*X_0^*$. Since $H_0^{-1} = H_0^*$, H_0 can be pulled out of the generalized inverse and therefore

$$Q_0 = \mathbf{I} - X_0(X_0^*X_0)^- X_0^* \, . \tag{62}$$

Incidentally, this equality shows that Q_0 is also the operator of orthogonal projection onto the orthogonal in $\mathbf{C}^U$ of the subspace generated by the columns of X_0. Then

$$E(y) = X_0H_0\xi_0 + Q_0X_1H_1\zeta_1 \tag{63}$$

where:

$$\xi_0 = \zeta_0 + (H_0^*X_0^*X_0H_0)^- H_0^*X_0^*X_1H_1\zeta_1 \tag{64}$$

In the reparametrized model (63), the two parts of the incidence matrix are orthogonal: $(X_0H_0)^{\mathrm{t}}Q_0X_1H_1 = 0$. Hence, a least squares estimate of ζ_1 is given by:

$$\hat{\zeta}_1 = (H_1^*X_1^*Q_0X_1H_1)^- H_1^*X_1^*Q_0\, y \tag{65}$$

We let:

$$\Omega_1 = H_1^*X_1^*Q_0X_1H_1 \tag{66}$$

Ω_1 is called the *information matrix* for ζ_1. A non null linear function $a^{\mathrm{t}}\zeta_1$ is estimable if and only if $a \in \operatorname{Im}\Omega_1$, or equivalently if there exists a b such that $a = \Omega_1 b$. In that case, it is estimated by $a^{\mathrm{t}}\hat{\zeta}_1$ with variance $\operatorname{var}(a^{\mathrm{t}}\hat{\zeta}_1) = a^{\mathrm{t}}\Omega^- a\,\sigma^2$.

It can be convenient, in order to compare designs of different sizes with the same error variance σ^2, to divide the information matrix by the number $N = |U|$ of units, thus getting a *per-unit information matrix* Ω_1/N and a *per-unit variance* $Na^{\mathrm{t}}\Omega_1^- a\,\sigma^2$. It is easy to check that in a complete factorial design the per-unit information matrix is the identity matrix and consequently the per-unit variance of the contrast $a^{\mathrm{t}}\hat{\zeta}_1$ is $a^{\mathrm{t}}a\,\sigma^2$. Comparing then the design under study with a complete factorial one on the basis of the per-unit variance, we obtain the following definition of the *factor efficiency* $\operatorname{eff}(a)$ for estimating $a^{\mathrm{t}}\zeta_1$:

$$\begin{aligned} \operatorname{eff}(a) &= \frac{a^{\mathrm{t}}a}{Na^{\mathrm{t}}\Omega_1^- a} && \text{if } a \in \operatorname{Im}\Omega_1, \\ \operatorname{eff}(a) &= 0 && \text{if } a \notin \operatorname{Im}\Omega_1. \end{aligned} \tag{67}$$

Thus the efficiency is by definition 0 if $a^t\zeta_1$ is not estimable. The efficiency can be greater than 1. For instance, in a study of a three-level factor, assume that two levels only are experimented, each on one half of the experimental units. The contrast comparing these two levels is then estimated with efficiency 3/2. However the contrasts involving the third level are not estimable so the design is a bad one. More generally, it can be shown that even if the efficiency is greater than 1 for some contrast belonging to the factorial effect studied, any suitable measure of overall efficiency for the whole factorial effect will be smaller than 1 (see for example Kobilinsky, 1990).

Let $\lambda_1 \geq \cdots \geq \lambda_q > 0 = \lambda_{q+1} = \cdots = \lambda_p$ be the eigenvalues of Ω_1/N, each repeated according to the dimension of its eigenspace (p is the dimension of Ω_1). Let $a_1, \ldots, a_p$ be corresponding orthonormal eigenvectors. The contrasts $a_1^t\zeta_1, \ldots, a_p^t\zeta_1$ are called *principal contrasts* for the factorial effect generated by ζ_1 (or θ_1). The corresponding efficiencies, which are precisely $\lambda_1, \ldots, \lambda_p$ are called the *principal efficiencies* for the factorial effect. The words *basic*, or *canonical* have also been used in the literature to refer to these contrasts or eigenvalues (Pearce *et al.* 1974, John 1987).

The following proposition shows that the principal efficiencies can be computed directly from the complex linear model matrix X, exactly as if it were real. The nullity of many elements in X^*X may then considerably simplify calculations.

Proposition 6.1 *The principal efficiencies for the factorial effect generated by θ_1 are the eigenvalues of the per-unit information matrix Δ_1 for θ_1, defined by: $\Delta_1 = X_1^*Q_0X_1/N$, where $N = |U|$ is the total number of units and Q_0 the operator given in* (62).

The proposition follows immediately from the relations $\Omega_1/N = H_1^*\Delta_1 H_1$ and $H_1^*H_1 = \mathbf{I}$.

From each eigenvector b of Δ_1, one can deduce an eigenvector H_1^*b of Ω_1/N. However, H_1^*b is not necessarily real. To get real eigenvectors, the following proposition can be used.

Proposition 6.2 *If b is an eigenvector of Δ_1 with eigenvalue λ, H_1^*b and $\overline{H_1^*b}$ are eigenvectors of Ω_1/N with the same eigenvalue λ.*

Proof. $(\Omega_1/N)H_1^*b = H_1^*\Delta_1 b = \lambda H_1^*b$. Then: $(\Omega_1/N)\overline{H_1^*b} = \overline{(\Omega_1/N)}\ \overline{H_1^*b} = \overline{\lambda}\ \overline{H_1^*b} = \lambda\overline{H_1^*b}$ (since Ω_1/N and λ are real). □

If H_1^*b is not real, any vector $a = \alpha H_1^*b + \overline{\alpha}\ \overline{H_1^*b}$ is a real eigenvector of Ω_1 with eigenvalue λ. The corresponding principal contrast is: $a^*\zeta_1 = \overline{\alpha}b^*\theta_1 + \alpha\overline{b^*\theta_1}$.

Estimation and tests can also be obtained directly from X. For instance, consider a contrast $b^*\theta_1$. It is real iff the coordinates in b associated with conjugate parameters in θ_1 are conjugate (see Proposition 3.2). It is then estimable if $b \in \operatorname{Im}\Delta_1$, and in that case, its least squares estimate is $b^*\hat{\theta}_1$ where:

$$\hat{\theta}_1 = (X_1^*Q_0X_1)^- X_1^*Q_0 y \ . \tag{68}$$

Exactly as the global information matrix, the information matrix Δ_1 for θ_1 can be computed directly from (50) with weights proportional to the $|U_k|$. This is easily deduced from the matricial form (53) of (50) which becomes

$$E(W^*y) = W^*X_0\theta_0 + W^*X_1\theta_1 \qquad \text{var}(W^*y) = \sigma^2 W^*W \tag{69}$$

The matrix W^*W is diagonal with the $1/|U_k|$ on the diagonal. Since when the $|U_k|$ are not equal it is not proportional to the identity, the metric $(W^*W)^{-1}$ must be introduced in the computation. Orthogonality is with respect to this metric. The orthogonal projector Q_0 onto $(\text{Im}\, W^*X_0)^\perp$ and the per-unit information matrix Δ_1 for θ_1 are then

$$Q_0 = \mathbf{I} - W^*X_0(X_0^*W(W^*W)^{-1}W^*X_0)^- X_0^*W(W^*W)^{-1} \tag{70}$$

$$\Delta_1 = X_1^*W(W^*W)^{-1}Q_0W^*X_1/N \tag{71}$$

Using the equality $(W^*W)^{-1} = W^{-1}W^{*-1}$, it is easy to check that this Δ_1 deduced from (50) is the same as the one given in proposition 6.1.

In the particular case where all U_k have the same number n of units, $W^*W = \mathbf{I}/n$. If we let $Z_0 = W^*X_0$, $Z_1 = W^*X_1$, the following familiar expressions are obtained:

$$Q_0 = \mathbf{I} - Z_0(Z_0^*Z_0)^- Z_0^* \tag{72}$$

$$\Delta_1 = \frac{n}{N} Z_1^*Q_0Z_1 \tag{73}$$

7 Juxtaposition of GMG Designs: some general formulae

The framework of Section 5.1 is now restricted by the following assumptions.

- All the groups U_k are isomorphic to the same group G.
- Φ_k is equal to a morphism Φ independent of k.
- If there are blocks, $\mathcal{J}(k) = k$ and $\mathbf{v}_k = 0$ for each k (except in the special case of two blocks in Example 8).

Thus the treatment $\mathbf{t}$ on unit $\mathbf{u} \in U_k$ is

$$\phi_k(\mathbf{u}) = \Phi\mathbf{u} + \mathbf{t}_k\ . \tag{74}$$

If there are blocks, the unit $\mathbf{u} \in U_k$ belongs to the blocks defined by the coordinates of $\Psi_k\mathbf{u} \in V_k$. Any partition into blocks is therefore nested within the partition into the U_k.

The set of experimented treatments is $\mathcal{T} = \bigcup_{k=1}^K(\mathbf{t}_k + \text{Im}\,\Phi)$. The smallest regular fraction containing it is $\mathbf{t}+T_0$, where $\mathbf{t}$ is any element of $\mathcal{T}$ and T_0 the subgroup generated by $\mathcal{T} - \mathbf{t}$.

It is convenient to decompose Φ in a canonical way: $\Phi = \Lambda \circ \Phi_0$, where Φ_0 is the morphism from G into T_0 coinciding with Φ and $\Lambda : T_0 \longrightarrow T$ the canonical injection.

The CEFs for the regular fraction $\mathbf{t}+T_0$ are the linear combinations $\gamma(B)$ associated with the characters B of T_0 defined by

$$\gamma(B) = \sum_{A:A \circ \Lambda = B} A(\mathbf{t})e(A) \ . \tag{75}$$

It is clear that any estimable function for the subfraction $\mathcal{T}$ of $\mathbf{t}+T_0$ is a linear combination of the $\gamma(B)$. More precisely, if there are no block effects, (50) gives

$$E(\langle y_k, C\rangle /|G|) = \sum_{A:A \circ \Phi = C} A(\mathbf{t}_k)e(A) = \sum_{B:B \circ \Phi_0 = C} \ \sum_{A:A \circ \Lambda = B} A(\mathbf{t}_k)e(A)$$

Since $\mathbf{t}_k - \mathbf{t} \in T_0$, $A(\mathbf{t}_k - \mathbf{t}) = A(\Lambda(\mathbf{t}_k - \mathbf{t})) = B(\mathbf{t}_k - \mathbf{t})$ whenever $A \circ \Lambda = B$. So

$$\sum_{A:A \circ \Lambda = B} A(\mathbf{t}_k)e(A) = \sum_{A:A \circ \Lambda = B} A(\mathbf{t}_k - \mathbf{t})A(\mathbf{t})e(A) = B(\mathbf{t}_k - \mathbf{t})\gamma(B)$$

and model (50) becomes

$$E(\langle y_k \, , \, C\rangle /|G|) = \sum_{B:B \circ \Phi_0 = C} B(\mathbf{t}_k - \mathbf{t})\gamma(B) \tag{76}$$

where the sum is over characters B confounded with C and such that $\gamma(B)$ is not zero by hypothesis.

When there are blocks, this equality remains valid if C is not confounded with blocks on U_k, that is if $C \notin \mathrm{Im}\, \Psi_k^*$. If on the contrary C is confounded, there is a character D of V_k such that $C = D \circ \Psi_k$ and, since Ψ_k is surjective, this character is unique. The factorial block effect $e_k(D)$ must then be added to the expectation in (76) which is replaced by

$$E(\langle y_k \, , \, C\rangle /|G|) = \sum_{B:B \circ \Phi_0 = C} B(\mathbf{t}_k - \mathbf{t})\gamma(B) + e_k(D) \ . \tag{77}$$

The block parameters appearing in distinct equations in (77) are distinct. The contrasts $\langle y_k \, , \, C\rangle /|G|$ such that C is confounded with blocks are therefore useless to estimate the treatment parameters $\gamma(B)$ and attention can be restricted to the equalities (76) where C is not confounded with blocks. Moreover since the subsets $\Phi_0^{*-1}(C)$ associated with the different unit characters C are disjoint, each subset of equalities in (76) associated with a given C can be handled separately.

So we now consider a given unit character C and the indices k such that C is not confounded with block effects on U_k. For simplicity, these indices are assumed to be the L first ones. The corresponding model can be written as

$$E(z) = Z\theta \qquad \mathrm{var}(z) = \frac{\sigma^2}{|G|}\mathbf{I} \ . \tag{78}$$

The vectors z, θ and the matrix Z are

$$z = \begin{bmatrix} \langle y_1, C\rangle /|G| \\ \vdots \\ \langle y_L, C\rangle /|G| \end{bmatrix} \qquad \theta = \begin{bmatrix} \gamma(B_1) \\ \vdots \\ \gamma(B_p) \end{bmatrix} \qquad Z = \begin{bmatrix} B_1(\mathbf{t}_1 - \mathbf{t}) & & B_p(\mathbf{t}_1 - \mathbf{t}) \\ \vdots & \cdots & \vdots \\ B_1(\mathbf{t}_L - \mathbf{t}) & & B_p(\mathbf{t}_L - \mathbf{t}) \end{bmatrix}$$

where $B_1, \ldots, B_p$ are the characters appearing on the right hand side of (76).

The information matrix for θ is $Z^*Z = (z_{ij})$, with

$$z_{ij} = \sum_{k=1}^{L} (\overline{B}_i B_j)(\mathbf{t}_k - \mathbf{t}) \ . \tag{79}$$

The equalities $B_i \circ \Phi_0 = C$, $B_j \circ \Phi_0 = C$ imply $(\overline{B}_i B_j) \circ \Phi_0 = \mathbf{1}$. Hence there is exactly one character A_{ij} of $T_0/\operatorname{Im}\Phi_0$ such that

$$\overline{B}_i B_j = A_{ij} \circ \Pi \tag{80}$$

where $\Pi : T_0 \longrightarrow T_0/\operatorname{Im}\Phi_0$ is the quotient canonical mapping and

$$z_{ij} = \sum_{k=1}^{L} A_{ij}(\Pi(\mathbf{t}_k - \mathbf{t})) \ . \tag{81}$$

Srivastava & Throop (1990) study conditions on the family $(\Pi(\mathbf{t}_k - \mathbf{t}))_k$ under which there is orthogonality, that is $z_{ij} = 0$ for $i \neq j$. But their examples of resolution 5 fractions, a 2^{6-1} and a 3/4 of a 2^8 are not very interesting from the practical point of view since regular resolution 5 fractions with an equal or even lower number of units exist for these cases.

In the non-orthogonal case, if we are interested only in the contrasts generated by some of the parameters, say the last $p-q$ ones, we write the expectation of z in partitioned form:

$$E(z) = Z_0\theta_0 + Z_1\theta_1 \tag{82}$$

where $\theta_0 = (\gamma(B_1), \ldots, \gamma(B_q))^{\mathrm{t}}$, $\theta_1 = (\gamma(B_{q+1}), \ldots, \gamma(B_p))^{\mathrm{t}}$ and $Z = (Z_0, Z_1)$ is the corresponding partition of Z. Then if $Q_0 = \mathbf{I} - Z_0(Z_0^*Z_0)^- Z_0^*$, the per-unit information matrix for θ_1 obtained from the equality (73) is

$$\Delta_1 = Z_1^* Q_0 Z_1/K = \left(Z_1^*Z_1 - Z_1^*Z_0(Z_0^*Z_0)^- Z_0^*Z_1\right)/K \ . \tag{83}$$

In practice the parameters of interest are the effects A in T^*. To be estimable, such an effect must be the only non-zero one in the corresponding sum (75), that is in its coset $A \operatorname{Ker}\Lambda^*$. The number p is the number of cosets of $\operatorname{Ker}\Lambda^*$ in $A \operatorname{Ker}\Phi^*$ including non-zero effects. These cosets are obtained by multiplying by A the cosets of $\operatorname{Ker}\Lambda^*$ in $\operatorname{Ker}\Phi^*$.

We now consider some more special cases.

7.1 Estimation with one missing coset

We go on studying the treatment effects confounded with a given character C itself unconfounded with blocks on the first L isomorphic groups $U_1, \ldots, U_L$.

Assumption 7.1 *Each coset of* $\operatorname{Im}\Phi$ *in* $\mathbf{t} + T_0$ *appears* R *times* ($R \geq 1$) *in the list* $\mathbf{t}_1 + \operatorname{Im}\Phi, \ldots, \mathbf{t}_L + \operatorname{Im}\Phi$ *except one which appears only* $R - 1$ *times.*

If $R = 1$, $\mathbf{t}_1 + \mathrm{Im}\,\Phi$, $\ldots$, $\mathbf{t}_L + \mathrm{Im}\,\Phi$ are thus all but one of the distinct cosets of $\mathrm{Im}\,\Phi$ in $\mathbf{t} + T_0$.

Let $\mathbf{t}_0 + \mathrm{Im}\,\Phi$ be the coset appearing only $R-1$ times. Then the sequence $\Pi(\mathbf{t}_0 - \mathbf{t})$, $\ldots$, $\Pi(\mathbf{t}_K - \mathbf{t})$ contains R times each element of $T_0/\,\mathrm{Im}\,\Phi$ and

$$\sum_{k=0}^{L} A_{ij}(\Pi(\mathbf{t}_k - \mathbf{t})) = R\,\langle A_{ij}, \mathbf{1}\rangle \ .$$

The last scalar product is 0 if $A_{ij} \neq 1$, that is if $i \neq j$ and $L+1$ if $i = j$. But

$$z_{ij} = \sum_{k=0}^{L} A_{ij}(\Pi(\mathbf{t}_k - \mathbf{t})) - A_{ij}(\Pi(\mathbf{t}_0 - \mathbf{t}))$$

and so, if δ_{ij} is the Kronecker symbol,

$$z_{ij} = \delta_{ij}(L+1) - (\overline{B}_i B_j)(\mathbf{t}_0 - \mathbf{t}) \ . \tag{84}$$

Let now

$$h_0 = (B_1(\mathbf{t}_0 - \mathbf{t}), \ldots, B_q(\mathbf{t}_0 - \mathbf{t})) \qquad h_1 = (B_{q+1}(\mathbf{t}_0 - \mathbf{t}), \ldots, B_p(\mathbf{t}_0 - \mathbf{t})) \ . \tag{85}$$

It follows from (84) that

$$Z_0^* Z_0 = (L+1)\mathbf{I} - h_0^* h_0 \qquad Z_1^* Z_0 = -h_1^* h_0 \qquad Z_1^* Z_1 = (L+1)\mathbf{I} - h_1^* h_1 \ . \tag{86}$$

Since $p \leq |\,\mathrm{Ker}\,\Phi_0^*| = (L+1)/R$, we have $q < L+1$. Then $Z_0^* Z_0$ is invertible and

$$(Z_0^* Z_0)^{-1} = \frac{1}{L+1}\left[\mathbf{I} + \frac{h_0^* h_0}{L+1-q}\right] \ . \tag{87}$$

Replacing $(Z_0^* Z_0)^{-}$, $Z_1^* Z_0$, $Z_1^* Z_1$ in (83) by the above values and using the equality $h_0 h_0^* = q$ gives

$$\Delta_1 = \frac{L+1}{K}\left(\mathbf{I} - \frac{h_1^* h_1}{L+1-q}\right) \ .$$

Then

$$\Delta_1 h_1^* = \frac{L+1}{K}\,\frac{L+1-p}{L+1-q}\,h_1^*$$

$$\Delta_1 h^* = \frac{L+1}{K} h^* \qquad \text{for any } (p-q) \times 1 \text{ vector } h^* \text{ orthogonal to } h_1^*.$$

The corresponding eigenvalues are:

$$\frac{L+1}{K}\frac{L+1-p}{L+1-q} \qquad \text{with multiplicity 1}$$

$$\frac{L+1}{K} \qquad \text{with multiplicity } p-q-1$$

These results are summed up in the following proposition

	p	1	2	3	4
efficiency	$\frac{4}{3}\frac{4-p}{5-p}$	1	$\frac{8}{9}$	$\frac{2}{3}$	0

Table 6: *Efficiencies in John's three-quarter replicate.*

Proposition 7.1 *Let $\gamma(B_1)$, ..., $\gamma(B_p)$ be the non-zero CEFs of the regular fraction $\mathbf{t} + T_0$ which are confounded by Φ with the character C of G. Let L be the number of indices $k \in \{1, \ldots, K\}$ such that C is not confounded with blocks on U_k. Assume that, among the corresponding subsets $\mathbf{t}_k + \operatorname{Im}\Phi$ of treatments, each coset of $\operatorname{Im}\Phi$ in $\mathbf{t} + T_0$ appears R times except one which appears $R - 1$ times.*

Then the principal efficiencies to estimate $(\gamma(B_{q+1}), \ldots, \gamma(B_p))$ are $(L+1)/K$ with multiplicity $p - q - 1$ and $[(L+1)/K][(L+1-p)/(L+1-q)]$ with multiplicity 1.

7.1.1 $K/(K+1)$ fraction of a regular fraction

In this case $R = 1$, $L = K$ and $\mathbf{t}_1 + \operatorname{Im}\Phi$, ..., $\mathbf{t}_K + \operatorname{Im}\Phi$ are all but one of the distinct cosets of $\operatorname{Im}\Phi$ in $\mathbf{t} + T_0$. Block factorial effects if any must only be confounded with the $\gamma(B)$ which are not to be estimated. The principal efficiencies given in proposition 7.1 become $(K+1)/K$ (multiplicity $p - q - 1$) and $[(K+1)/K][(K+1-p)/(K+1-q)]$ (multiplicity 1).

For a 2/3 fraction and $p = 2$, $q = 1$, the proposition thus gives the efficiency 3/4, in agreement with the result of Example 6.

The proposition can also be used to get efficiencies in John's *three-quarter replicates.* Lists of these useful $3(2^{s-r})$ fractions, first studied by P.W.M. John (1962, 1971), can be found in Diamond (1981) and McLean & Anderson (1984). For them $T = (2)^s$, T_0 is a resolution 4 or 5 subgroup of size 2^{s-r+2} and G a subgroup of index 4 (size 2^{s-r}) of T_0. Thus $K = 3$ and $p - q = 1$. The factor efficiency for a single canonical estimable function $\gamma(B)$ confounded with $p - 1$ other ones on G depends only on p and is given in Table 6. A CEF on T_0 is therefore also estimable after omitting a coset of G iff there is at least one zero CEF among the four confounded with it on G.

Example 8 . For instance let T_0 be the subgroup of $(2)^6$ of resolution 4 defined by $E = ABC$, $F = ABD$, and G be the subgroup of T_0 defined by $C = A$, $D = A$. Then the CEF confounded with $\mathbf{1}$ on T_0 is $\gamma(\mathbf{1}) = e(\mathbf{1}) + e(ABCE) + e(ABDF) + e(CDEF)$ and the four CEFs on G are

$$\begin{aligned}
\delta(\mathbf{1}) &= \gamma(\mathbf{1}) + \gamma(AC) + \gamma(AD) + \gamma(CD)\\
\delta(A) &= \gamma(A) + \gamma(C) + \gamma(D) + \gamma(ACD)\\
\delta(B) &= \gamma(B) + \gamma(ABC) + \gamma(ABD) + \gamma(BCD)\\
\delta(AB) &= \gamma(AB) + \gamma(BC) + \gamma(BD) + \gamma(ABCD)
\end{aligned}$$

2 blocks :	$\mathcal{J}(k) = 1$	$\Psi_k = AB$
3 blocks :	$\mathcal{J}(k) = k$	$\Psi_k = \mathbf{1}$
6 blocks :	$\mathcal{J}(k) = k$	$\Psi_k = AB$

Table 7: *Three way of splitting into blocks a* $3(2^{6-4})$ *of resolution* 4

If it is assumed that interactions between 3 factors or more are zero, we have

$$\begin{array}{lll} e(A) = \gamma(A), & e(B) = \gamma(B), & e(C) = \gamma(C), \\ e(D) = \gamma(D), & e(E) = \gamma(ABC), & e(F) = \gamma(ABD), \end{array}$$

and moreover

$$\begin{array}{rcl} \gamma(ACD) & = & e(ACD) + e(BDE) + e(BCF) + e(AEF) \;\; = \;\; 0 \\ \gamma(BCD) & = & e(BCD) + e(ADE) + e(ACF) + e(BEF) \;\; = \;\; 0 \end{array}$$

Hence the 6 main effects are estimable with efficiency 2/3 in any design including 3 among the 4 cosets of G in some coset of T_0. Any such $3(2^{6-4})$ fraction is therefore of resolution 4.

Moreover main effects appear only in $\delta(A)$ and $\delta(B)$. It is therefore possible to divide the experiment into 2, 3 or 6 blocks without losing the resolution 4. The corresponding definition of $\mathcal{J}$ and Ψ_k are given in Table 7.

When there are no blocks, the two contrasts between the three means $\langle y_k, \mathbf{1} \rangle /4$ and the three contrasts $\langle y_k, AB \rangle /4$ can be used to get an over-estimate of σ^2. This estimate may be swollen by the presence of two-factor interactions, but this does not really matter if the aim of the experiment is to detect only main effects which are significantly greater than interactions.

With 6 blocks, these 5 contrasts whose expectation include 5 different block effects cannot be used to get an error variance. With 3 blocks, only the last three ones can be used. With 2 blocks, the expectations of the three contrasts $\langle y_k, \mathbf{1} \rangle /4$ include the same block effect $e_1(\mathbf{1})$ with coefficient 1. Thus $e_1(\mathbf{1})$ disappears from the expectation of the two orthonormal contrasts

$$\langle y_1, \mathbf{1} \rangle - \langle y_2, \mathbf{1} \rangle$$

$$\langle y_1, \mathbf{1} \rangle + \langle y_2, \mathbf{1} \rangle - 2 \langle y_3, \mathbf{1} \rangle$$

which can therefore be used to get an error variance biased only by interaction terms. Similarly two other independent contrasts whose expectations include only interactions can be deduced from the three contrasts $\langle y_k, AB \rangle /4$.

7.1.2 Macroblock designs

When the experiments must be blocked, the sets $U_1, \ldots, U_K$ provide a partition into *macroblocks* which can be further divided by means of the morphisms Ψ_k. We assume here that the whole design is made up of R replicates of the regular fraction T_0 (each divided into K/R macroblocks) and that the subgroups $\mathrm{Im}\,\Psi_k^*$ of G have only $\mathbf{1}$ as common character. Therefore a character $C \neq 1$ of G is confounded with blocks in at most one macroblock and proposition 7.1 applies with $L = K - 1$ when C is confounded.

The efficiency for a parameter $\gamma(B)$ is 0 if $B \circ \Phi_0 = \mathbf{1}$, is 1 if $C = B \circ \Phi_0$ is never confounded with blocks. If C is confounded with blocks on one U_k ($C \in \operatorname{Im} \Psi_k^*$), the principal efficiencies for any set of $p - q$ out of the p non-zero parameters $\gamma(B_1)$, ..., $\gamma(B_p)$ confounded with C on G are 1 with multiplicity $p - q - 1$ and $(K - p)/(K - q)$ with multiplicity 1.

In Example 1, $K = 2$. If the three-factor interactions are zero, $p = 1$, $q = 0$ and proposition 7.1 directly gives the efficiency of 1/2 for the 6 two-factor interactions confounded with blocks in one of the macroblocks. In Example 2, solution 2, $K = 4$. If no hypothesis is made on interactions between three or more factors, $p = 2$, $q = 1$. Proposition 7.1 then gives the efficiency 2/3 for the factorial effects confounded with blocks in one macroblock (Table 1). If interactions of three factors or more are zero, $p = 1$, $q = 0$ and the efficiency 3/4 found then for these effects can also be deduced from the proposition.

We give another example with factors at 3 levels, then show how to use a three-quarter replicate to get an efficient macroblock design.

Example 9 . 3^5 in 27 blocks of size 9.

The treatments are divided into three macroblocks according to the value of $ABCDE$. Each macroblock is then identified with $G = (3)^4$ with A, B, C, D as basic factors. The subsequent divisions into 9 blocks of the macroblocks use the following pairs of characters of G:

ACD, BC^2D	for the macroblock	$ABCDE = 1$,
ACD^2, BCD	for the macroblock	$ABCDE = \mathrm{j}$,
AC^2D, BC^2D^2	for the macroblock	$ABCDE = \mathrm{j}^2$.

It is easy to check that the subgroups generated by these pairs of characters have only $\mathbf{1}$ in common. Note that to find those subgroups , G was identified with the vectorial space F_9^2 over the Galois field F_9 of order 9. The subgroups were selected as three of the $10 = (81 - 1)/(9 - 1)$ one-dimensional subspaces.

Table 8 gives the factorial effects confounded together and with a block effect in the first macroblock. It contains no main effect and only two 2-factor interactions BE, B^2E^2. A similar result holds for the two other macroblocks. The other confounded 2-factor interactions are AE, A^2E^2 in macroblock 2 and DE, D^2E^2 in macroblock 3. The factor efficiency is 0 for the factorial effects in $\operatorname{Ker} \Phi^* = \{\mathbf{1}, ABCDE, A^2B^2C^2D^2E^2\}$. It is 1 for main effects and more generally for all factorial effects which are never confounded with blocks.

Consider now one of the 2-factor interactions confounded with blocks in one macroblock, say BE. It is also confounded on each macroblock with AB^2CDE^2 and $A^2C^2D^2$. If the 5-factor interactions are assumed to be 0, proposition 7.1 applies with $K = L+1 = 3$, $p = 2$, $q = 1$ and the efficiency is found to be 1/2. If 3-factor interactions are also assumed to be 0, then $p = 1$, $q = 0$ and the efficiency becomes 2/3. The same result holds for the

1	$ABCDE$	$A^2B^2C^2D^2E^2$
ACD	$A^2BC^2D^2E$	B^2E^2
$A^2C^2D^2$	BE	AB^2CDE^2
BC^2D	AB^2D^2E	A^2CE^2
ABD^2	A^2B^2CE	C^2DE^2
A^2BC	B^2C^2DE	AD^2E^2
B^2CD^2	AC^2E	A^2BDE^2
AB^2C^2	A^2DE	BCD^2E^2
A^2B^2D	CD^2E	ABC^2E^2

Table 8: *Treatment effects confounded with a block effect in macroblock* 1.

five other 2-factor interactions confounded with blocks in one macroblock.

Any 3-factor interaction with equal exponents like ACD is confounded with one 2-factor interaction and one 5-factor interaction. If it is confounded with blocks in a macroblock, it is therefore estimable with efficiency 1/2 if 5-factor interactions are assumed to be 0. Finally each 3-factor interaction with unequal exponents like BC^2D is confounded with another 3-factor interaction and a 4-factor interaction. If it is confounded with blocks in a macroblock, it is estimable with efficiency 1/2 only if 4-factor interactions are also assumed to be 0.

Example 10 . 2^{12-4} in 32 blocks of size 8.

The tables of $3(2^{s-r})$ fractions can be used to divide the corresponding regular fractions T_0 (isomorphic to 2^{s-r+2}) into "small" blocks, instead of reducing the size of the design by a factor 3/4. The four quarters are taken as macroblocks and divided as indicated at the beginning of the section. Any effect which would be estimable in the 3/4 fraction is also estimable in the block design since it is confounded in at most one macroblock and thus estimable from at least three macroblocks.

As an example, consider the 3/64 fraction of 2^{12} factorial given in McLean & Anderson (1984, appendix 4, Table 9). The 12 factors are represented by the first letters of the alphabet, excluding I which is reserved by McLean and Anderson for the identity, but keeping G though it is also used to denote the group to which each macroblock is isomorphic. The regular fraction T_0 is defined by

$$J = -CFGH\ , \quad K = -ACEG\ , \quad L = -BDEFGH\ , \quad M = -CDEGH.$$

It is divided into 4 macroblocks using the values of the characters $ABCDG$, $CDEFH$. Each macroblock is identified with $G = (2)^6$ with A, B, C, D, E, F as basic factors. The divisions into 2^3 blocks of the macroblocks use the following sets of characters of G:

ADE,	BEF,	CD	for the macroblock	$(ABCDG, CDEFH) = (\ 1,\ 1)$
ADF,	$BDEF$,	CDE	for the macroblock	$(ABCDG, CDEFH) = (-1,\ 1)$
AEF,	BD,	CE	for the macroblock	$(ABCDG, CDEFH) = (\ 1, -1)$
$ADEF$,	BDE,	CEF	for the macroblock	$(ABCDG, CDEFH) = (-1, -1)$

$\mathbf{1}$	$ABCDG$	$CDEFH$	$ABEFGH$
$\mathbf{1}$	$ABCDG$	$CDEFH$	$ABEFGH$
$CDEGHM$	$ABEHM$	$\underline{\underline{FGM}}$	$ABCDFM$
$BDEFGHL$	$ACEFHL$	$\underline{BCGL}$	$\underline{\underline{ADL}}$
$BCFLM$	$ADFGLM$	$BDEHLM$	$ACEGHLM$
$ACEGK$	$\underline{BDEK}$	$ADFGHK$	$BCFHK$
$ADHKM$	$BCGHKM$	$ACEFKM$	$BDEFGKM$
$ABCDFHKL$	$FGHKL$	$ABEKL$	$CDEGKL$
$ABEFGKLM$	$CDEFKLM$	$ABCDGHKLM$	$\underline{HKLM}$
$CFGHJ$	$ABDFHJ$	$\underline{DEGJ}$	$ABCEJ$
$DEFJM$	$ABCEFGJM$	$\underline{CHJM}$	$ABDGHJM$
$BCDEJL$	$AEGJL$	$BFHJL$	$ACDFGHJL$
$BGHJLM$	$ACDHJLM$	$BCDEFGJLM$	$AEFJLM$
$AEFHJK$	$BCDEFGHJK$	$ACDJK$	$\underline{BGJK}$
$ACDFGJKM$	$BFJKM$	$AEGHJKM$	$BCDEHJKM$
$ABDGJKL$	$\underline{CJKL}$	$ABCEFGHJKL$	$DEFHJKL$
$ABCEHJKLM$	$DEGHJKLM$	$ABDFJKLM$	$CFGJKLM$

Table 9: *Example 10. Characters confounded with* **1** *on G*

The subgroups generated by these couples have only **1** in common. They can be obtained as 4 out of the 9 one-dimensional subspaces of F_8^2, where F_8 is the field of polynomials modulo $1 + x^2 + x^3$, and treatment $(t_1, t_2, t_3, t_4, t_5, t_6)$ is identified with $(t_1 + t_2x + t_3x^2, t_4 + t_5x + t_6x^2)$.

Table 9 gives the 64 treatment effects confounded with **1** on G. Their arrangement in columns is such that each column is the subset $\Lambda^{*-1}(\xi)$ of treatment characters confounded on T_0 with a character $\xi \in \operatorname{Ker} \Phi_0^*$. This character ξ is given on top of the column. In particular the first column gives the treatments characters confounded with **1** on T_0. Except **1**, all of them have at least five letters and T_0 has therefore resolution 5. If interactions of three or more factors are zero, the non-zero CEFs of T_0 are precisely the main effects and two-factor interactions.

Assume that all interactions involving 3 factors or more are 0. The non-zero effect confounded on G with a given main effect can only come from the multiplication by **1** and by the two 3-letter effects doubly underlined in Table 9. Since these two 3-letter effects have no letter in common, a main effect is therefore confounded with at most $p = 2$ non-zero CEFs of T_0 (including itself) and is estimated with minimum efficiency $(4-2)/(4-1) = 2/3$. Similarly the non-zero effects confounded with a two-factor interaction can come only from multiplication by **1** and by the 9 underlined 3 or 4-letter interactions. Since T_0 is of resolution 5, there is at most one such non-zero effect for each column of the table and the examination of the intersections between underlined effects in different columns shows that a two-factor interaction can be confounded at most with $p = 3$ characters of T_0 hence estimated with minimum efficiency $1/2$.

The non-zero effects confounded with blocks on each macroblock are given in Ta-

	*macroblock*1						
character of G	ADE	BEF	$ABDF$	CD	ACE	$BCDEF$	$ABCF$
treatment character	EL		$CM,\ HJ$	CD	$GK,\ BJ$	BH	DM
efficiency	3/4		2/3	3/4	2/3	3/4	3/4
	*macroblock*2						
character of G	ADF	$BDEF$	ABE	CDE	$ACEF$	BCF	$ABCD$
treatment character	FL	FK	$HM,\ KL,\ CJ$	FH	$HL,\ KM$	$LM,\ HK$	$G,\ FM$
efficiency	3/4	3/4	1/2	3/4	2/3	2/3	2/3
	*macroblock*3						
character of G	AEF	BD	$ABDEF$	CE	ACF	$BCDE$	$ABCDF$
treatment character		$BD,\ EK$		CE		$CK,\ JL$	$FG,\ M$
efficiency		2/3		3/4		2/3	2/3
	*macroblock*4						
character of G	$ADEF$	BDE	ABF	CEF	ACD	$BCDF$	$ABCE$
treatment character		K		DH	$BG,\ CL,\ JK$	AM	J
efficiency		3/4		3/4	1/2	3/4	3/4

Table 10: *Example 10. Non zero effects confounded on each macroblock*

ble 10 with the corresponding efficiencies. The other non-zero effects are estimated with efficiency 1.

7.2 Two-coset fractions: $K = 2$.

The information matrix Δ_1 for θ_1 given in (83) is invertible if and only if the $p-q$ columns of $Q_0 Z_1$ are independent. But, when $K = 2$, these columns are in $\mathbf{C}^2$ and moreover, when $q \geq 1$, they are in a subspace of $\mathbf{C}^2$ of dimension ≤ 1. So Δ_1 is invertible and θ_1 estimable only if $p - q = 1$ or $q = 0$, $p = 2$.

- Case $p = 2$, $q = 0$. Then

$$\Delta_1 = Z_1^* Z_1/2 = \begin{bmatrix} 1 & \alpha \\ \overline{\alpha} & 1 \end{bmatrix}$$

  where (taking $\mathbf{t} = \mathbf{t}_2$)

$$\alpha = z_{12}/2 = [(\overline{B_1} B_2)(\mathbf{t}_1 - \mathbf{t}_2) + 1]/2 \,. \tag{88}$$

  The eigenvalues of Δ_1 are $1 + |\alpha|$, $1 - |\alpha|$ and its determinant $|\Delta_1| = 1 - |\alpha|^2$. The latter is 0 iff $|\alpha| = 1$, equivalently iff $(\overline{B_1} B_2)(\mathbf{t}_1 - \mathbf{t}_2) = 1$. The maximum global efficiency as measured by $|\Delta_1|$ is obtained when the argument of $(\overline{B_1} B_2)(\mathbf{t}_1 - \mathbf{t}_2)$ is as near of π as possible.

- Case $p - q = 1$, $q = 1$. The efficiency for $\gamma(B_2)$ is $(Z_1^* Z_1 - Z_1^* Z_0 (Z_0^* Z_0)^{-1} Z_0^* Z_1)/2 = 1 - |\alpha|^2$.

- Case $p - q = 1$, $q = 0$. The efficiency is 1.

Consider again Example 7. Since the product $\overline{B}_1 B_2$ belongs to Ker Φ, it is equal to $(AB)^i(CDEF)^k$ and $(\overline{B}_1 B_2)(\mathbf{t}_1 - \mathbf{t}_2) = (-1)^i \mathrm{j}^{2k}$. Moreover if B_1 and B_2 are main effects or two-factor interactions, either i or k is 0, hence $|\alpha|^2 = 0$ or $|\alpha|^2 = 1/4$, which leads to the two efficiencies 1 and 3/4 previously found.

References

Bailey R.A. (1982). Dual abelian groups in the design of experiments. In: *Algebraic Structures and Applications* (Eds P.Schultz, C.E.Praeger and R.P.Sullivan), 45-54. Marcel Dekker, New York.

Chakravarti I.M. (1976). Optimal linear mapping of a Burnside group and its applications. Atti Dei Convegni Lincei 17, Colloquio Internazinale sulle Teorie Combinatorie (Roma 3-15 settembre 1973), Tomo I, 171-181.

Connor W.S. & Young S. (1961). Fractional factorial designs with factors at two and three levels. National Bureau of Standards. Applied Mathematics Series. **58**. (Reproduced in McLean & Anderson, 1984, Marcel Dekker, New-York .

Diamond W.J. (1981). Practical Experiment Design for Engineers and Scientists. Lifetime Learning Publications, Belmont.

El Mossadeq A., Kobilinsky A. & Collombier D. (1985). Construction d'orthogonaux dans les groupes abéliens finis et confusions d'effets dans les plans factoriels. *Linear Algebra Applic.* **70**, 303-320.

El Mossadeq A. & Kobilinsky A. (1992). Run orders and quantitative factors in asymmetrical designs. *Applied Stochastic Models and Data Analysis*, **8**, 259-281.

John J.A. (1973). Generalized cyclic designs in factorial experiments. *Biometrika* **60**, 55-63.

John J.A. (1987). Cyclic Designs. Chapman and Hall.

John, P.W.M. (1962). Three quarter replicates of 2^n designs. *Biometrics*, **18**, 172-184.

John P.W.M. (1971). Statistical Design and Analysis of Experiments. New-York: Macmillan.

Kempthorne O. (1952). Design and Analysis of Experiments. (2 nd edition 1973). Ed: Krieger.

Kobilinsky A. (1985). Confounding in relation to duality of finite abelian groups. *Linear Algebra Applic.* **70**, 321-347.

Kobilinsky A. (1990). Complex linear models and cyclic designs. *Linear Algebra Applic.* **127**, 227-282.

Kobilinsky A. & Monod H. (1991). Experimental design generated by group morphisms: an introduction. *Scand. J. Statist.*, **18**, 119-134.

McLean R. & Anderson V.L. (1984). Applied Factorial and Fractional Designs. Ed: Dekker.

Patterson H.D. (1976). Generation of factorial designs. *J. Roy. Statist. Soc. Ser. B*, **38**, 175-179.

Pearce S.C., Calinski T. & Marshall T.F. (1974). The basic contrasts of an experimental design with special reference to the analysis of data. *Biometrika*, **61**, 449-460.

Srivastava J.N. & Throop D. (1990). Orthogonal Arrays Obtainable as Solutions to Linear Equations over Finite Fields. *Linear Algebra Applic.* **127**, 283-300.

Winer B.J.(1962). Statistical principles in experimental design. (2nd edition 1971). McGraw-Hill.

Yates, F.(1937). The design and analysis of factorial experiments. Tech. Commonw. Bur. Soil Sci. No. 35, Harpenden.

Received March 1993, in final form May 1994

A. Kobilinsky
Laboratoire de Biomtrie
Route de Saint-Vyr
F 78026 VERSAILLES Cedex
FRANCE
email : kobi@versailles.inra.fr