Complex linear models and cyclic designs

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ABSTRACT Many orthogonal factorial designs can be defined by abelian group morphisms. By juxtaposition of such designs, useful non-orthogonal designs can also be obtained, including the classical generalized cyclic designs, as well as a new kind of one replicate factorial block designs. Their efficiencies are easily computed by means of a complex reparametrization based on the irreducible characters of the groups involved. The theory extends to the "group generated" designs defined by Bailey and Rowley [4], in which the group is not necessarily abelian. In some cases, we give explicit formulas for the efficiencies of these latter designs.

Key words: abelian group, construction of block designs, cyclic design, design of experiments, dual group, efficiency factor, generalized cyclic design, multiplicity free character, permutation character, permutation group.

1 Introduction

The abelian group theory may be used to construct many orthogonal fractional replicates or block designs in confounding, as described in Bailey [1, 2, 3], and Kobilinsky [11, 12]. In these designs, Abelian group structures are given to the set of experimental units, the set of treatments and , for block designs, the set of blocks. The mapping assigning a treatment, or a block, to each experimental unit is then chosen among affine mappings, which are the composition of a group morphism and a translation.

Using a complex reparametrization, associated with the irreducible characters of the groups involved, a very simple description of aliasing can be given. One of the purposes of this paper is to show how the same type of reparametrization can be used to obtain the efficiency factors for the contrasts of interest in some more complicated situations, in which the set of units is identified with a disjoint union of abelian groups and the mapping assigning the treatments (resp. blocks) is affine on each of these groups. Generalized cyclic designs [8] can be brought in that context. For them, the complex information matrix is block diagonal. Each block involves only one treatment effect and the corresponding efficiency factor is quite easy to derive.

With this complex reparametrization, we also analyse another new scheme, which gives a useful method of blocking for single replicate factorial designs (section 7). Indeed, with two or more replicates, it is possible to balance the loss of information due to the blocks over the replicates, by confounding different sets of degrees of freedom in the different replicates (see Bose [5]). But with only one replicate, even if there is a restrictive model on treatment effects, total confounding is generally inadequate. The method presented here gives then an alternative which leads to a certain balance in the loss of information. Its principle is to realize the division into blocks in two steps. In the first step, only negligible effects (generally high order interactions) are confounded. This step gives a limited number of *macro-blocks*. In the subsequent division of macro-blocks into blocks, the loss of information is balanced over the effects of interest by confounding different sets of degrees of freedom in the different macro-blocks.

Both generalized cyclic designs and the preceding scheme are particular cases of the group generated block designs studied by Bailey and Rowley [4]. It is therefore natural to devote part of this paper to these block designs (section 5,6). In fact, there is a way to obtain some of these designs which parallels the process described at the beginning of this introduction : the set of units is identified with the set G/A of left cosets of a subgroup A of the finite group G. The mapping assigning a treatment, or a block, to each experimental unit is defined by the canonical surjection from G/A onto the set of left cosets G/B of a subgroup B containing A. We shall show that any of the group generated block designs of Bailey and Rowley (BR) can be generated by juxtaposing such simple designs. After having reparametrized with complex parameters as in the abelian case, we will then generalize an explicit formula for the efficiency factor of treatment contrasts given by BR in the abelian case to designs based on arbitrary groups (but satisfying a criterion given by BR which guarantees their general balance). In that formula appears an irreducible character of the group. When this character is linear (it is always the case with abelian groups), we will see that it is in fact possible to give a simpler formula.

Before studying these designs, we will introduce some results about the complex model (section 3). We will see that its use can notably simplify the calculations leading to the estimates and their variances, as well as the demonstration of optimality of certain orthogonal designs (section 4 and 6.5).

2 Preliminaries

We first recall some notations and results about abelian group generated design, in order to motivate the following developments.

Suppose there are ν crossed factors, having m_1, m_2, ..., m_ν levels respectively. The levels of the ith factor are labeled by the elements 0, 1, ..., $m_i - 1$ of the cyclic group of order m_i, denoted by C_i. The set of $m = m_1 \times \ldots \times m_\nu$ treatments can then be represented by the product group $T = C_1 \times \ldots \times C_\nu$. The elements of T are the n-tuples $t = (t_1, \ldots, t_\nu)$, where $t_i \in C_i$, and its addition is defined componentwise:

$$(t_1, \ldots, t_\nu) + (t'_1, \ldots, t'_\nu) = (t_1 + t'_1, \ldots, t_\nu + t'_\nu) \ .$$

The vector $\boldsymbol{\tau}$ of treatment effects is an element of the real vector space $\mathbb{R}^T$ of dimension m, which is naturally imbedded in the complex vector space $\mathbb{C}^T$. $\mathbb{C}^T$ is equipped with the usual scalar product defined by:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{t}} \overline{\mathbf{y}} \tag{2.1}$$

A useful reparametrization is obtained by decomposing τ on the orthogonal basis of $\mathbb{C}^T$ constituted by the *irreducible characters* of the group T [12, 14, 17] ([14] calls them *simple characters*). To give explicitly these characters, we let $T^\times = C_1 \times \ldots \times C_\nu$ be a group isomorphic to T and define a duality between T and $T^\times$ by

$$\left[\mathbf{t}^\times, \mathbf{t}\right] = \sum_{i=1}^{\nu} t_i^\times t_i M/m_i \quad (\text{modulo } M) \tag{2.2}$$

where $\mathbf{t} = (t_1, \ldots, t_\nu) \in T$, $\mathbf{t}^\times = (t_1^\times, \ldots, t_\nu^\times) \in T^\times$, and M is any common multiple of $m_1, \ldots, m_\nu$.

For η a primitive Mth root of unity (for instance $\eta = e^{2\pi \mathrm{i}/M}$), we associate to each $\mathbf{t} \in T^\times$ the vector $\boldsymbol{\eta}^{t^\times}$, the $\mathbf{t}^{\text{th}}$ coordinate of which is $\eta^{[\mathbf{t}^\times, \mathbf{t}]}$:

$$\boldsymbol{\eta}^{t^\times}(\mathbf{t}) = \eta^{[\mathbf{t}^\times, \mathbf{t}]} \tag{2.3}$$

These vectors are precisely the irreducible characters of T. They are orthogonal for the usual scalar product of $\mathbb{C}^T$ and they have the same square norm, which is the number $|T|$ of elements in T:

$$\left\langle \boldsymbol{\eta}^{\mathbf{t}^\times}, \boldsymbol{\eta}^{\mathbf{s}^\times} \right\rangle = 0 \quad \text{for } \mathbf{t}^\times \neq \mathbf{s}^\times, \quad \left\langle \boldsymbol{\eta}^{\mathbf{t}^\times}, \boldsymbol{\eta}^{\mathbf{t}^\times} \right\rangle = |T| \,. \tag{2.4}$$

The decomposition of $\boldsymbol{\tau}$ on the basis $(\boldsymbol{\eta}^{\mathbf{t}^\times})$ can be written:

$$\boldsymbol{\tau} = \sum_{\mathbf{t}^\times \in T^\times} \alpha_{\mathbf{t}^\times} \, \boldsymbol{\eta}^{\mathbf{t}^\times}, \tag{2.5}$$

where:

$$\alpha_{\mathbf{t}^\times} = \frac{\left\langle \boldsymbol{\tau}, \boldsymbol{\eta}^{\mathbf{t}^\times} \right\rangle}{|T|} \tag{2.6}$$

The nature of the complex parameter $\alpha_{\mathbf{t}^\times}$ can be immediately deduced from the non zero elements in $\mathbf{t}^\times = (t_1^\times, \ldots, t_\nu^\times)$. For instance:

1. if $\mathbf{t}^\times = (0, \ldots, 0)$, then $\alpha_{\mathbf{t}^\times}$ is the general mean:

$$\alpha_{\mathbf{t}^\times} = \sum_{t_1, \ldots, t_\nu} \frac{\boldsymbol{\tau}(t_1, \ldots, t_\nu)}{|T|}$$

2. if $\mathbf{t}^\times = (t_1^\times, 0, \ldots, 0)$ where $t_1^\times \neq 0$, then $\alpha_{\mathbf{t}^\times}$ is a contrast between the marginal means for the different levels of factor 1, hence belongs to the main effect of factor 1:

$$\alpha_{\mathbf{t}^\times} = \sum_{t_1} \eta^{-t_1^\times t_1 M/m_1} \sum_{t_2,\ldots,t_\nu} \frac{\boldsymbol{\tau}(t_1,\ldots,t_\nu)}{|T|}$$

3. if $\mathbf{t}^\times = (t_1^\times, t_2^\times, 0, \ldots, 0)$ with $t_1^\times \neq 0$, $t_2^\times \neq 0$, then $\alpha_{\mathbf{t}^\times}$ belongs to the interaction between factors 1 and 2, and so on.

The subsets of elements $\mathbf{t}^\times$ associated to a given effect (we include under this denomination interactions as well as main effects) are thus easy to identify. An important property of these subsets, which will be often used later, is that they are stable for the operation $\mathbf{t}^\times \mapsto -\mathbf{t}^\times$.

In many circumstances, some effects are assumed to be zero, so that the real vector τ satisfies the relation :

$$\boldsymbol{\tau} = \sum_{\mathbf{t}^\times \in S^\times} \alpha_{\mathbf{t}^\times} \boldsymbol{\eta}^{\mathbf{t}^\times} \tag{2.7}$$

where $S^\times$ includes all $\mathbf{t}^\times$ associated to non null effects. It must be noticed that $S^\times$ contains the opposite of any of it elements, and that the parameters associated to opposite elements of $S^\times$ are conjugated: $\alpha_{\mathbf{t}^\times} = \overline{\alpha_{-\mathbf{t}^\times}}$. If $\mathbf{t}^\times = -\mathbf{t}^\times$, $\alpha_{\mathbf{t}^\times}$ is thus a real parameter. But if $\mathbf{t}^\times \neq -\mathbf{t}^\times$, $\alpha_{\mathbf{t}^\times}$ and $\alpha_{-\mathbf{t}^\times}$ are complex, hence cannot receive any interpretation. We could replace them by two real parameters of the same nature, as it is done in [12]. However calculations are easier if estimation is carried on directly on complex parameters and results on real parameters subsequently derived. Indeed, any real linear form of $\boldsymbol{\tau}$ can be expressed as a linear form of the vector $\boldsymbol{\alpha} = (\alpha_{\mathbf{t}^\times})_{\mathbf{t}^\times} \in S^\times$ of complex parameters. The point, made precised in the next section, is then that the least squares estimate of that linear form and its variance can be derived from the model written as a function of $\boldsymbol{\alpha}$ exactly as if $\boldsymbol{\alpha}$ were real. The normal equations in this last model, which will be called a complex linear model, can be very simple, and the calculations consequently quite simplified.

To write the model more precisely , suppose that the experimental design is defined by the function $\mathbf{u} \mapsto \phi(\mathbf{u})$ assigning to each experimental unit $\mathbf{u} \in U$ the associated treatment in T. Suppose also that the expectation of the explained variable y on unit $\mathbf{u}$, denoted by $y(\mathbf{u})$, depends only of the treatment $\phi(\mathbf{u})$: $E(y(\mathbf{u})) = \boldsymbol{\tau}(\phi(\mathbf{u}))$. Using (2.7) and (2.3), we can express this expectation as a function of the parameters $\alpha_{\mathbf{t}^\times}$:

$$E(y(\mathbf{u})) = \sum_{\mathbf{t}^\times \in S^\times} \alpha_{\mathbf{t}^\times} \eta^{[\mathbf{t}^\times, \phi(\mathbf{u})]} \tag{2.8}$$

In matrix notation, this becomes :

$$E(\mathbf{y}) = X\boldsymbol{\alpha} \tag{2.9}$$

where :

$\mathbf{y} = (y(\mathbf{u}))_{\mathbf{u}\in U}$ is the $|U| \times 1$ vector of observations,

$X = \left(\eta^{[\mathbf{t}^\times, \phi(\mathbf{u})]}\right)_{\mathbf{u}\in U, \mathbf{t}^\times \in S^\times}$ the design matrix

$\boldsymbol{\alpha} = (\alpha_{\mathbf{t}^\times})_{\mathbf{t}^\times \in S^\times}$ the vector of complex parameters.

The stability of $S^\times$ under the operation $\mathbf{s}^\times \mapsto -\mathbf{s}^\times$ insures that the parameters in $\boldsymbol{\alpha}$ and corresponding columns in X are either real, or conjugated in pairs. This is the only property which will be needed in the development of the next section.

When ϕ is a group morphism, X has a specially simple structure. Let U be decomposed as a product of κ cyclic groups and M be an exponent of the two groups U and T (i.e. a common multiple of the orders of the cyclic groups in the decompositions of U and T). ϕ can be represented by an $\nu \times \kappa$ matrix. There is a dual morphism, from $T^\times$ into $U^\times$ (see [12]) whose $\kappa \times \nu$ matrix $\phi^\times$ satisfies:

$$\forall \mathbf{t} \in T, \forall \mathbf{t}^\times \in T^\times, \quad \left[\mathbf{t}^\times, \phi\mathbf{u}\right] = \left[\phi^\times \mathbf{t}^\times, \mathbf{u}\right] \tag{2.10}$$

($\phi\mathbf{u}$, the product of the matrix ϕ with the vector $\mathbf{u} = (u_1, \dots, u_\kappa)'$, is equal to the image $\phi(\mathbf{u})$ of $\mathbf{u} \in U$ by the morphism ϕ). The column of index $\mathbf{t}^\times$ in X is then: $\left(\eta^{[\mathbf{t}^\times, \phi\mathbf{u}]}\right)_{\mathbf{u}\in U}$ $= \left(\eta^{[\phi^\times \mathbf{t}^\times, \mathbf{u}]}\right)_{\mathbf{u}\in U} = \eta^{\phi^\times \mathbf{t}^\times}$. Thus the columns of indices $\mathbf{t}_1^\times$ and $\mathbf{t}_2^\times$ are either equal if $\phi^\times \mathbf{t}_1^\times = \phi^\times \mathbf{t}_2^\times$, or orthogonal.

In the situation described above, there are ν crossed factors respectively associated with the components of the product $T = C_1 \times \dots \times C_\nu$. Other situations where there are nesting relations between factors can be handled in a similar way. The set of treatments is also represented by a product $T = C_1 \times \dots \times C_\nu$ of cyclic groups. The levels of a factor can be defined by the values of any morphism defined on T, and not necessarily by the values of the coordinates on $C_1, \dots, C_\nu$. The parameters $\alpha_{\mathbf{t}^\times}$ are also of interest in these situations, and their nature (type of effect) can be deduced from $\mathbf{t}^\times$. We refer to Kobilinsky [12] for a more detailed account.

3 The complex linear model

3.1 The model

We consider the linear model :

$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\alpha}, \quad \text{var}(\mathbf{y}) = \sigma^2 \mathbf{I_n} \tag{3.1}$$

where $\mathbf{y}$ is the $n \times 1$ vector of observations, X the $n \times p$ design matrix, which has complex coefficients, and $\boldsymbol{\alpha}$ the $p \times 1$ vector of complex parameters.

We suppose, as it is the case in (2.9), that the parameters in $\boldsymbol{\alpha}$ and corresponding columns in X are either real or conjugate in pairs. More precisely, let J be a set indexing the different parameters and denote by α_j the j th coordinate of $\boldsymbol{\alpha}$ and by $\mathbf{x}_j$ the j th column of X. The above hypothesis amounts to the existence of an operation $j \mapsto -j$ in J, satisfying $-(-j) = j$, such that:

$$\forall j \in J, \quad \mathbf{x}_{-j} = \overline{\mathbf{x_j}} \tag{3.2}$$

$$\forall j \in J, \quad \alpha_{-j} = \overline{\alpha_j} \tag{3.3}$$

To simplify notations, the set of vectors $\boldsymbol{\alpha} = (\alpha_j)$ satisfying (3.3) will be denoted by Θ:

$$\Theta = \{\boldsymbol{\alpha} | \forall j \in J, \ \ \alpha_{-j} = \overline{\alpha}_j\}. \tag{3.4}$$

(3.3) can be replaced by the apparently weaker following statement (supposing (3.2) true) :

$$X\boldsymbol{\alpha} \quad \text{is a real vector.} \tag{3.5}$$

It is clear that (3.3) implies (3.5). Conversely, if $X\boldsymbol{\beta}$, where $\boldsymbol{\beta} \in \mathbb{C}^p$ is a real vector, then it is possible to find $\boldsymbol{\alpha}$ satisfying (3.3), i.e. belonging to Θ, such that $X\boldsymbol{\alpha} = X\boldsymbol{\beta}$. To find this $\boldsymbol{\alpha}$, we split the terms $\beta_j \mathbf{x}_j$ with $j = -j$, and the partial sums $\beta_j \mathbf{x}_j + \beta_{-j}\mathbf{x}_{-j}$ with $j \neq -j$, into their real and imaginary components. $X\boldsymbol{\beta}$ being real is equal to the sum of the real components which are :

- for $j = -j$:

$$\mathcal{R}\left(\beta_j \mathbf{x}_j\right) = \alpha_j \mathbf{x}_j, \qquad \text{where } \alpha_j = \mathcal{R}\left(\beta_j\right)$$

- for $j \neq -j$:

$$\begin{aligned}
\mathcal{R}\left(\beta_j \mathbf{x}_j + \beta_{-j}\mathbf{x}_{-j}\right) &= \frac{\left(\beta_j \mathbf{x}_j + \beta_{-j}\mathbf{x}_{-j}\right) + \left(\overline{\beta}_j \overline{\mathbf{x}}_j + \overline{\beta}_{-j}\overline{\mathbf{x}}_{-j}\right)}{2} \\
&= \frac{\left(\beta_j + \overline{\beta}_{-j}\right)\mathbf{x}_j + \left(\beta_{-j} + \overline{\beta}_j\right)\mathbf{x}_{-j}}{2} = \alpha_j \mathbf{x}_j + \alpha_{-j}\mathbf{x}_{-j}
\end{aligned}$$

where

$$\alpha_j = \frac{\beta_j + \overline{\beta}_{-j}}{2} \quad \text{and} \quad \alpha_{-j} = \frac{\beta_{-j} + \overline{\beta}_j}{2}\ .$$

The α_j defined above are the coordinates of the sought vector $\boldsymbol{\alpha}$.

The expectation model can be written in a more geometrical form :

$$E(\mathbf{y}) \in \mathbf{E} \cap \mathbb{R}^n \tag{3.6}$$

where $\mathbf{E}$ is the subspace $\operatorname{Im} X$ of $\mathbb{C}^n$, which is self-conjugated:

$$\mathbf{E} = \overline{\mathbf{E}}\ . \tag{3.7}$$

The following proposition gives an interesting property of such a subspace, which will be used in section 6.

Proposition 3.1 *Any self-conjugate subspace of $\mathbb{C}^n$ has a real basis.*

Proof. Let $\mathbf{E}$ be a self-conjugate subspace of $\mathbb{C}^n$. We form by recurrence a basis of $\mathbf{E}$ which belongs to $\mathbb{R}^n$. Suppose that the k first vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$ of this basis have been chosen, where $k < \dim \mathbf{E}$. Let $\mathbf{x}$ be a vector of $\mathbf{E}$ outside the space $\mathbf{H}$ generated by $\mathbf{e}_1, \ldots, \mathbf{e}_k$ in $\mathbb{C}^n$. Then $\mathbf{x} + \overline{\mathbf{x}}$ and $i\mathbf{x} - i\overline{\mathbf{x}}$ are two real vectors of $\mathbf{E}$ generating the same subspace as $\mathbf{x}$ and $\overline{\mathbf{x}}$. One of them does not belongs to $\mathbf{H}$ and can be chosen as $\mathbf{e}_{k+1}$ ■

3.2 Least Squares Estimate

The least squares estimate of $E(\mathbf{y})$ under model (3.6) is the orthogonal projection of $\mathbf{y}$ on the subspace $\mathbf{E} \cap \mathbb{R}^n$ of $\mathbb{R}^n$. The next proposition states that this projection is identical to the orthogonal projection of $\mathbf{y}$ on the subspace $\mathbf{E}$ in $\mathbb{C}^n$.

Proposition 3.2 *Let P be the operator of orthogonal projection on a self-conjugate subspace $\mathbf{E}$ of $\mathbb{C}^n$. Then the restriction of P to $\mathbb{R}^n$ is the operator of orthogonal projection on $\mathbf{E} \cap \mathbb{R}^n$.*

Proof. For every $\mathbf{y} \in \mathbb{R}^n$, $P\mathbf{y}$ is the point of $\mathbf{E}$ which is closest to $\mathbf{y}$, and hence such that:

$$\|\mathbf{y} - P\mathbf{y}\|^2 = \min_{\mathbf{x} \in \mathbf{E}} \|\mathbf{y} - \mathbf{x}\|^2$$

The following equalities then show that $\overline{P\mathbf{y}}$ is the point in $\overline{\mathbf{E}}$ closest to $\mathbf{y}$, that is the orthogonal projection of $\mathbf{y}$ on $\overline{\mathbf{E}}$.

$$\|\mathbf{y} - \overline{P\mathbf{y}}\|^2 = \|\mathbf{y} - P\mathbf{y}\|^2 = \min_{\mathbf{x} \in \mathbf{E}} \|\mathbf{y} - \mathbf{x}\|^2 = \min_{\mathbf{x} \in \mathbf{E}} \|\mathbf{y} - \overline{\mathbf{x}}\|^2 = \min_{\mathbf{x} \in \overline{\mathbf{E}}} \|\mathbf{y} - \mathbf{x}\|^2$$

and therefore, $\overline{P\mathbf{y}}$ is the point in $\overline{\mathbf{E}}$ closest to $\mathbf{y}$, i.e.the orthogonal projection of $\mathbf{y}$ on $\overline{\mathbf{E}}$. As $\mathbf{E} = \overline{\mathbf{E}}$, we have $\overline{P\mathbf{y}} = P\mathbf{y}$. Thus Py belongs to $\mathbb{R}^n$ and is the orthogonal projection of $\mathbf{y}$ on $\mathbf{E} \cap \mathbb{R}^n$ ■

Let P be the operator of orthogonal projection on the subspace $\mathbf{E} = \operatorname{Im} X$ of $\mathbb{C}^n$. If $\hat{\boldsymbol{\alpha}}$ is a vector in $\mathbb{C}^p$ such that $Py = X\hat{\boldsymbol{\alpha}}$, we shall say, as in the real case, that $\hat{\boldsymbol{\alpha}}$ is a least squares estimate of $\boldsymbol{\alpha}$. From the orthogonality between $\mathbf{y} - P\mathbf{y}$ and $\operatorname{Im} X$, the following normal equations, where $X^* = \overline{X}'$, are immediately obtained:

$$X^* X \boldsymbol{\alpha} = X^* \mathbf{y} \tag{3.8}$$

If $(X^*X)^-$ is a generalized inverse of X^*X, we have :

$$P\mathbf{y} = X\hat{\boldsymbol{\alpha}} = \left[X(X^*X)^-(X^*X)\right] \hat{\boldsymbol{\alpha}} = X(X^*X)^- X^* \mathbf{y}$$

hence the expression of P as a function of X is :

$$P = X(X^*X)^- X^* \; . \tag{3.9}$$

3.3 Real reparametrization

To get the preceding results, it is also possible to use a real reparametrization. For each $\boldsymbol{\alpha} \in \Theta$, we define a real vector of parameters $\boldsymbol{\beta}$ by the equality:

$$\boldsymbol{\beta} = N\boldsymbol{\alpha} \tag{3.10}$$

where N is an invertible matrix such that, for each j, the columns of indices j and $-j$ are conjugated. An example of such a matrix is given in [12].

Proposition 3.3 *The mapping $\boldsymbol{\alpha} \mapsto \boldsymbol{\beta}$ is a one to one correspondance between Θ and $\mathbb{R}^p$.*

Proof. If $\boldsymbol{\alpha} \in \Theta$, $N\boldsymbol{\alpha}$ clearly belongs to $\mathbb{R}^p$. Moreover, the mapping $\boldsymbol{\alpha} \mapsto N\boldsymbol{\alpha}$ is injective. The surjectivity follows from the following lemma, which implies that $N^{-1}\boldsymbol{\beta}$ belongs to Θ for every $\boldsymbol{\beta} \in \mathbb{R}^p$ ■

Lemma 3.1 *Let N be a square invertible matrix whose rows and columns are indexed by the elements of J. If the columns (resp. rows) of indices j and $-j$ are conjugate for every $j \in J$, the rows (resp. columns) of N^{-1} of indices j and $-j$ are conjugate for every $j \in J$.*

Proof. Let R be the matrix of the permutation of J which exchange j and $-j$ for every pair of distinct opposite elements j, $-j$ of J. The product matrix NR is deduced from N by exchanging the columns associated to opposite elements, which are conjugated by hypothesis. Hence $NR = \overline{N}$, and consequently :

$$R^{-1}N^{-1} = \overline{N}^{-1} = \overline{N^{-1}}$$

It follows from the last equality that in N^{-1} the rows of indices j and $-j$ are conjugates ■

The expectation model in 3.1) can thus be written :

$$E(\mathbf{y}) = XN^{-1}\boldsymbol{\beta} \tag{3.11}$$

where $\boldsymbol{\beta}$ is a vector of $\mathbb{R}^p$ (and XN^{-1} a real matrix). The operator of orthogonal projection on XN^{-1} is:

$$P = XN^{-1}(N^{*-1}X^*XN^{-1})^{-}N^{*-1}X^* = XN^{-1}\left[N(X^*X)^{-}N^*\right]N^{*-1}X^* = X(X^*X)^{-1}X^*$$

A least square estimate of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = N\boldsymbol{\alpha}$, where $\hat{\boldsymbol{\alpha}}$ is given by (3.8), and so on.

3.4 Estimability

A linear form $_alphab \mapsto < \mathbf{a}, \boldsymbol{\alpha} >$ on $\mathbb{C}^p$ will be said to be estimable in the model (3.1) if it admits an unbiased estimate $< \mathbf{b}, \mathbf{y} >$, in other words if there exists a vector $\mathbf{b} \in \mathbb{C}^n$ such that :

$$\forall \boldsymbol{\alpha} \in \Theta, \quad < \mathbf{a}, \boldsymbol{\alpha} >=< \mathbf{b}, X\boldsymbol{\alpha} > \tag{3.12}$$

It is easy to see that the subset Θ generates $\mathbb{C}^p$. Since $< \mathbf{b}, X\boldsymbol{\alpha} >=< X^*\mathbf{b}, \boldsymbol{\alpha} >$, (3.12) implies that $\mathbf{a} = X^*\mathbf{b}$. Conversely, if $\mathbf{a} = X^*\mathbf{b}$ for a $\mathbf{b} \in \mathbb{C}^n$, then (3.12) is satisfied. Hence $< \mathbf{a}, \boldsymbol{\alpha} >$ is estimable if and only if $\mathbf{a} \in \operatorname{Im} X^*$.

Moreover, we have: $< \mathbf{b}, X\boldsymbol{\alpha} >=< P\mathbf{b}, X\boldsymbol{\alpha} >=< X(X^*X)^- X^*\mathbf{b}, X\boldsymbol{\alpha} >=< X\mathbf{c}, X\boldsymbol{\alpha} >$, with $c = (X^*X)^- X^*\mathbf{b}$. Hence, the estimability of $< \mathbf{a}, \boldsymbol{\alpha} >$ is also equivalent to the existence of a vector $\mathbf{c} \in \mathbb{C}^p$ such that :

$$\forall \boldsymbol{\alpha} \in \Theta, \quad < \mathbf{a}, \boldsymbol{\alpha} >=< X\mathbf{c}, X\boldsymbol{\alpha} > \tag{3.13}$$

It must be noticed that $X\mathbf{c}$ is uniquely defined, even if $\mathbf{c}$ is not.

A linear form $< \mathbf{a}, \boldsymbol{\alpha} >$ of the parameters will be said to be real if it is real for every $\boldsymbol{\alpha} \in \Theta$. An equivalent condition is that $\mathbf{a} \in \Theta$, i.e. that the coordinates a_j of $\mathbf{a}$ associated to opposite elements of J are conjugate:

$$\forall j \in J, \quad a_j = \overline{a}_{-j} \tag{3.14}$$

This last condition clearly implies that $< \mathbf{a}, \boldsymbol{\alpha} >$ is real for every $\boldsymbol{\alpha} \in \Theta$. Conversely, suppose that $< \mathbf{a}, \boldsymbol{\alpha} >$ is real for every $\boldsymbol{\alpha} \in \Theta$. By taking $\boldsymbol{\alpha}$ such that $\alpha_j = 1$, $\alpha_{-j} = 1$ and $\alpha_k = 0$ for each $k \neq j$, we see that $a_j + a_{-j}$ is real. Taking then $\boldsymbol{\alpha}$ such that $\alpha_j = \mathrm{i}$, $\alpha_{-j} = -\mathrm{i}$ and $\alpha_k = 0$ for each $k \neq j$, we see that $\mathrm{i}(a_j - a_{-j})$ is also real, and condition (3.14) follows.

A necessary and sufficient condition for $< \mathbf{a}, \boldsymbol{\alpha} >$ to be simultaneously estimable and real is that there exists a *real* $\mathbf{b}$ such that $\mathbf{a} = X^*\mathbf{b}$. If $\mathbf{b}$ is real, $< \mathbf{a}, \boldsymbol{\alpha} >=< \mathbf{b}, X\boldsymbol{\alpha} >$ is real for every $\boldsymbol{\alpha} \in \Theta$, since $X\boldsymbol{\alpha}$ is then real. Conversely, suppose $< \mathbf{a}, \boldsymbol{\alpha} >=< \mathbf{b}, X\boldsymbol{\alpha} >$ is real for every $\boldsymbol{\alpha} \in \Theta$ and let $\mathbf{b} = \mathbf{b}_1 + \mathrm{i}\mathbf{b}_2$ be the decomposition of $\mathbf{b}$ into its real and imaginary part. We have $< \mathbf{b}_2, X\boldsymbol{\alpha} >= 0$ for $\boldsymbol{\alpha} \in \Theta$, hence $X^*\mathbf{b}_2 = 0$ and finally $\mathbf{a} = X^*\mathbf{b}_1$.

Moreover, if $< \mathbf{a}, \boldsymbol{\alpha} >$ is estimable and real, we can choose the vector $\mathbf{c}$ of (3.13) to be in Θ, i.e. such that:

$$\forall j \in J \quad c_j = \overline{c}_{-j}(3.15) \tag{3.15}$$

(the demonstration is analogous to that of the equivalence between (3.3) and (3.5)).

In factorial design, one is concerned with the subsets of parameters corresponding to the different effects. In section 2, we saw that such subsets are associated with subsets of $T^\times$ which are stable for the operation $t^\times \mapsto -t^\times$. This leads us, in the more general context of model (3.1), to study effects associated to subsets of J which are stable for the operation $j \mapsto -j$.

So, let J_1 be a subset of J satisfying:

$$j \in J_1 \implies -j \in J_1 \tag{3.16}$$

By definition, the (real) *linear forms of the effect associated to* J_1 are the (real) linear combinations of parameters α_j for $j \in J_1$. Let us partition $\boldsymbol{\alpha}$ as $(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1)$, where $\boldsymbol{\alpha}_1$ is formed by the coordinates of index j in J_1, and X accordingly as (X_0, X_1). Then, the linear forms of this effect are the linear forms $< \mathbf{a}_1, \boldsymbol{\alpha}_1 >$. They are real if and only if the coordinates associated to opposite elements of J_1 in $\mathbf{a}_1$ are conjugate. The set of vectors $\mathbf{a}_1 = (a_j)$, $j \in J_1$, satisfying this last condition will be denoted Θ_1:

$$\Theta_1 = \{\mathbf{a}_1 | \forall j \in J_1,\ a_{-j} = \overline{a}_j.$$

A linear form $< \mathbf{a}_1, \boldsymbol{\alpha}_1 >$ is *estimable* if there exists $\mathbf{b} \in \mathbb{C}^n$ such that $< \mathbf{a}_1, \alpha_1 >=< \mathbf{b}, X\boldsymbol{\alpha} >$. Since $< \mathbf{b}, X\boldsymbol{\alpha} >=< \mathbf{b}, X_0\boldsymbol{\alpha}_0 > + < \mathbf{b}, X_1\boldsymbol{\alpha}_1 >$, we have $\mathbf{a}_1 = X_1^*\mathbf{b}$ and $0 = X_0^*\mathbf{b}$. Thus $\mathbf{b}$ is orthogonal to $\operatorname{Im} X_0$, and satisfies $\mathbf{b} = Q_0\mathbf{b}$, where Q_0 is the operator of orthogonal projection on the orthogonal complement of $\operatorname{Im} X_0$ in $\mathbb{C}^n$. Moreover, if $\mathbf{b}$ is chosen in $\operatorname{Im} X$ ($\mathbf{b} = X\mathbf{c}$), we have $\mathbf{b} = Q_0X\mathbf{c} = Q_0X_1\mathbf{c}_1$, and finally $\mathbf{a}_1 = X_1^*Q_0X_1\mathbf{c}_1$.

Conversely, if there exists $\mathbf{c}_1$ such that $\mathbf{a}_1 = X_1^*Q_0X_1\mathbf{c}_1$, $< \mathbf{a}_1, \boldsymbol{\alpha}_1 >$ is an estimable contrast of the effect associated to J_1. $\mathbf{b} = Q_0X_1\mathbf{c}_1$ is then the only vector in $\operatorname{Im} X$ such that $< \mathbf{a}_1, \boldsymbol{\alpha}_1 >=< \mathbf{b}, X\boldsymbol{\alpha} >$ for every $\boldsymbol{\alpha} \in \Theta$. The following proposition sums up the preceding results:

Proposition 3.4 *A linear form $< \mathbf{a}_1, \boldsymbol{\alpha}_1 >$ is estimable if and only if there exists $\mathbf{c}_1$ such that $\mathbf{a}_1 = X_1^*Q_0X_1\mathbf{c}_1$. It is real if and only if the coordinates of $\mathbf{a}_1$ associated to opposite elements of J_1 are conjugate, or equivalently if $\mathbf{c}_1$ can be chosen so that its coordinates associated to opposite elements of J_1 are conjugate.*

3.5 Estimation of linear form of the parameters

It is well known that the minimum variance estimate of a real linear form $< \mathbf{a}, \boldsymbol{\alpha} >=< \mathbf{b}, X\boldsymbol{\alpha} >$, where $\mathbf{b} \in \mathbb{R}^n$, is:

$$< a, \hat{\boldsymbol{\alpha}} >=< \mathbf{b}, X\hat{\boldsymbol{\alpha}} >=< \mathbf{b}, P\mathbf{y} >=< P\mathbf{b}, \mathbf{y} > \tag{3.17}$$

The variance of this estimate is:

$$\operatorname{var}(< \mathbf{a}, \hat{\boldsymbol{\alpha}} >) = \sigma^2 < P\mathbf{b}, P\mathbf{b} >= \sigma^2 < P\mathbf{b}, \mathbf{b} >= \sigma^2\mathbf{a}^*(X^*X)^-\mathbf{a} \tag{3.18}$$

Thus, the variance is given by a formula analogous to that used when the incidence matrix X is real. The only difference is the replacement of the transpose by the conjugate transpose. We don't have to worry about the fact that the expectation space is not the whole $\operatorname{Im} X$, but only the real part of it.

When $\mathbf{b} = X\mathbf{c}$, the estimate is $< \mathbf{b}, \mathbf{y} >$ and the variance $\sigma^2 < \mathbf{b}, \mathbf{b} >= \sigma^2\mathbf{c}^*(X^*X)\mathbf{c}$.

If $< \mathbf{a}, \boldsymbol{\alpha} >$ belongs to the effect associated to J_1, it can be written as $< \mathbf{a}_1, \boldsymbol{\alpha}_1 >$ with $\mathbf{a}_1 = X_1^*Q_0X_1\mathbf{c}_1$. Its least squares estimate is then $< Q_0X_1\mathbf{c}_1, \mathbf{y} >$ with variance $\sigma^2 < Q_0X_1\mathbf{c}_1, Q_0X_1\mathbf{c}_1 >= \sigma^2\mathbf{c}_1^*X_1^*Q_0X_1\mathbf{c}_1 = \sigma^2\mathbf{a}_1^*(X_1^*Q_0X_1)^-\mathbf{a}_1$. (the last equality follows directly from the definition of a generalized inverse).

Proposition 3.5 *Let* $< \mathbf{a}_1, \alpha_1 >$ *be a real estimable linear form of* $\boldsymbol{\alpha}_1$. *Its least squares estimate is* $< \mathbf{a}_1, \hat{\boldsymbol{\alpha}}_1 >$ *where* $\hat{\boldsymbol{\alpha}}_1 = (X_1^* Q_0 X_1)^- X_1^* Q_0 \mathbf{y}$. *The variance of this estimate is* $\sigma^2 \mathbf{a}_1^* (X_1^* Q_0 X_1)^- \mathbf{a}_1$.

The matrix $X_1^* Q_0 X_1$ will be called the information matrix for $\boldsymbol{\alpha}_1$.

4 Optimality and efficiency

4.1 Optimality

A good design is one which estimates the contrasts of interest with a variance as small as possible. Since it is generally not possible to get a minimal variance for all these contrasts, one usually defines a global measure of variance, such as the determinant, the trace, or the first eigenvalue of the matrix of covariance of a suitably chosen set of (real) parameters.

In factorial design, the different effects do not have the same importance, so that it is better, rather than to define an overall measure of variance, to consider first each effect separately.

So, we let J_1 be the subset of J, satisfying (3.16), associated to a given effect. X_0, X_1, $_alphab_0$, $\boldsymbol{\alpha}_1$, Q_0 are defined as at the end of section 3.4. Finally, we let q be the number of elements of J_1: $q = |J_1|$. Since the family of α_j, where $j \in J_1$, is stable by conjugation, a $q \times 1$ vector $\boldsymbol{\beta}_1$ of real parameters can be deduced from α_1 by an invertible transformation N having its columns associated to opposite parameters conjugate:

$$\boldsymbol{\beta}_1 = N \boldsymbol{\alpha}_1 \tag{4.1}$$

If all the parameters α_j for $j \in J_1$ are estimable, the matrix $X_1^* Q_0 X_1$ is invertible and the covariance matrix of the least squares estimator $\hat{\boldsymbol{\beta}}_1$ is $\sigma^2 V$, where:

$$V = N (X_1^* Q_0 X_1)^{-1} N^* \tag{4.2}$$

In this paper, we shall only compare designs with the same error variance σ^2, and shall therefore no longer take this parameter into account.

Since it is sometimes desirable to compare designs with different numbers of experimental units, we multiply V by the number n of units to get a *per unit* covariance matrix. We can then use the logarithm of the determinant $[\log \det(nV)]$, the trace $[\mathrm{trace}(nV)]$ or the first eigenvalue $[\lambda_{\max}(nV)]$ as a measure of global variance for this effect. If $X_1^* Q_0 X_1$ is not invertible, we can define this measure to be $+\infty$ since such a situation is clearly undesirable if we are equally interested by all the contrasts of the given effect.

The preceding measures are maybe more clearly defined from the eigenvalues λ_1, ... , λ_q of the per unit information matrix:

$$C = \frac{N^{*-1} (X_1^* Q_0 X_1) N^{-1}}{n} \tag{4.3}$$

which is the inverse of nV when V exists. If we adopt the convention that $1/0 = +\infty$, we can write them:

$$\psi_0(C) = \sum \log \frac{1}{\lambda_i} \quad [\text{for } \log\det(nV)\ ] \tag{4.4}$$

$$\psi_1(C) = \sum \frac{1}{\lambda_i} \quad [\text{for trace}(nV)] \tag{4.5}$$

$$\psi_2(C) = \max \frac{1}{\lambda_i} \quad [\text{for } \lambda_{\max}(nV)] \tag{4.6}$$

Kiefer [10] gave a useful tool to find in some situations the best design(s) with respect to the above measures of variance. That tool works in fact for a larger family of functions ψ of C: ψ is any function defined on the set $\mathcal{M}$ of symmetric positive matrices of dimension $q \times q$, with values in $]-\infty, +\infty[$, which satisfies:

(a) ψ is convex: $\psi[\alpha A + (1-\alpha)B] \leq \alpha\psi(A) + (1-\alpha)\psi(B)$ for $0 \leq \alpha \leq 1$
(b) $\psi(\alpha A)$ is non increasing in the scalar α: $\alpha_1 \leq \alpha_2 \Longrightarrow \psi(\alpha_1 A) \geq \psi(\alpha_2 A)$
(c) ψ is orthogonally invariant: $\psi(N'AN) = \psi(A)$ for every orthogonal N.

(c) is equivalent to:

(c') $\psi(A)$ depends on A only through the eigenvalues of A.

If $\mathcal{D}$ is the set of matrices C associated to the different designs under consideration, and β is an upper bound of trace$(C)/q$, i.e:

$$\beta \geq \sup_{C \in \mathcal{D}} \frac{\text{trace}(C)}{q} \tag{4.7}$$

we have:

Proposition 4.1 *$\psi(C) \geq \psi(\beta \mathbf{I}_q)$ for all $C \in \mathcal{D}$*

Proof. Let $t = trace(C)/q$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_q)$ the diagonal matrix with the eigenvalues of C on its diagonal. If ρ is any permutation of $1, \ldots, q$ and $\rho\Lambda$ the matrix obtained from Λ by permuting rows and columns according to ρ, we have $\psi(\rho\Lambda) = \psi(\Lambda)$ by property (c). Let then G be a transitive group of permutations of $1, \ldots, q$ ($G = S_q$ for instance) and $\overline{\Lambda} = \left(\sum_{\rho \in G} \rho\Lambda\right)/|G|$. We clearly have: $\overline{\Lambda} = t\mathbf{I}_q$. It follows then from properties (a), (b) and (c) that:

$$\psi(C) = \psi(\Lambda) = \frac{\sum_{\rho \in G} \psi(\rho\Lambda)}{|G|} \geq \psi(\overline{\Lambda}) \geq \psi(\beta \mathbf{I}_q)$$

■

Corollary 4.1 *If $C_0 = \beta \mathbf{I}_q$ belongs to $\mathcal{D}$, the corresponding design is optimal with respect to any ψ satisfying (a), (b), (c).*

Such a design is called universally optimal in $\mathcal{D}$. Even when it does not exist, Proposition 4.1 gives a useful upper bound to $\psi(C)$ which can be taken as a reference to define efficiency.

This corollary is Kiefer's proposition 1'. In fact, Kiefer's result is a little more general. He establishes a one to one correspondance $C \mapsto \overline{C}$ which to each matrix $C \in \mathcal{D}$ associates a $(q+1) \times (q+1)$ matrix $\overline{C}$ with zero row and column sums. Instead of condition (c), he only requires that $\psi(C) = \omega(\overline{C})$ where ω is invariant for each permutation of rows and (the same on) columns of $\overline{C}$.

Still different conditions are given by Mukerjee [15], quoting Sinha and Mukerjee [18]. Instead of (c), they require the weaker condition that ψ itself is invariant for each permutation of rows and (the same on) columns. They then replace (b) by:

(b') $\psi(\alpha \mathbf{I}_p + \beta \mathbf{1}\mathbf{1}') \geq \psi(a\mathbf{I}_p)$ whenever $a \geq \alpha + \beta$.

Proposition 4.2 *Suppose that the matrix N in (4.3) is unitary and that* $\mathrm{trace}(X_1^* X_1/(nq)) \leq \beta$ *for any of the designs considered. We then have* $\mathrm{trace}(C)/q \leq \beta$ *for every $C \in \mathcal{D}$, hence $\psi(C) \geq \psi(\beta \mathbf{I}_q)$ for every ψc satisfying (a), (b) and (c).*

N is unitary iff $N^*N = \mathbf{I}_q$. A simple example of unitary N is obtained by taking $\beta_j = \alpha_j$ if $j = -j$ and $\beta_j = (\alpha_j + \alpha_{-j})/\sqrt{2}$, $\beta_{-j} = (-i\alpha_j + i\alpha_{-j})/\sqrt{2}$ if $j = -j$. (see [12]).

Proof.

$$\begin{aligned} n\,\mathrm{trace}(C) &= \mathrm{trace}\,[N^{*-1}(X_1^* Q_0 X_1)N^{-1}] = \mathrm{trace}\,[(X_1^* Q_0 X_1)N^{-1}N^{*-1}] \\ &= \mathrm{trace}\,[X_1^* Q_0 X_1] \leq \mathrm{trace}\,[X_1^* X_1] \leq nq\beta\ . \end{aligned}$$

The first inequality stems from the same inequality for diagonal elements: if $\mathbf{x}$ is a column of X_1, the inequality for the corresponding diagonal elements is $\|Q_0\mathbf{x}\|^2 \leq \|\mathbf{x}\|^2$. ∎

In model (2.9), all the elements of X, hence of X_1, are of modulus 1, whatever the function ϕ defining the design is. Therefore we have $\mathrm{trace}\,[X_1^* X_1/(nq)] = 1$, and the proposition applies with $\beta = 1$. To get then a design with corresponding matrix $C_0 = \mathbf{I}_q$, we must have:

(1) $Q_0 X_1 = X_1$, which means that the columns of X_1 are orthogonal to the other columns of X.

(2) $X_1^* X_1 = n\mathbf{I}_q$. In the context of section 2, this last condition is satisfied if the factor associated to the considered effect has equireplicate levels (for an interaction, we mean the product factor naturally associated).

Call a design equireplicate if each of the factors in the model has equireplicate levels. With orthogonality defined as in Tjur [19], we have:

Proposition 4.3 *If X has the form given in (2.9) and N is unitary, an equireplicate orthogonal design is universally optimal for the estimation of any of the factorial effects.*

It should be noted that it is also optimal for the estimation of any subspace of real contrasts generated by a self-conjugate subset of parameters $\{\alpha_j,\ j \in J_1\}$.

Proposition 4.3 is the analogous, in Tjur's terminology, of a result expressed by Mukerjee [15] in term of orthogonal arrays.

When N is not unitary, we get a weaker result by replacing universal optimality by D-optimality. Indeed,

$$\begin{aligned}\psi_0(C) = \log\det(nV) &= \log\det\left[Nn(X_1^* Q_0 X_1)^{-1} N^*\right] \\ &= \log\left\{\det N \det\left[n(X_1^* Q_0 X_1)^{-1}\right] \det N^*\right\} \\ &= K + \log\det\left[n(X_1^* Q_0 X_1)^{-1}\right]\ ,\end{aligned}$$

where $K = \log(\det N \det N^*)$ is a constant independant of the design.

Another application of proposition 4.2 will be given in section 6.5.

4.2 Efficiency

Since we are mainly concerned, throughout this paper, with treatment contrasts – that is with linear forms $\mathbf{b}^*\boldsymbol{\tau}$ where $\mathbf{b}^*\mathbf{1} = 0$ – we shall speak of "contrast" rather than of "linear form of the treatment parameters", even when all the linear forms involved are not contrasts. Moreover, we shall use the word "efficiency" instead of "efficiency factor" (as defined in John [9]) since we always assume, when comparing different designs, that they have the same error variance.

The efficiency of estimation of a contrast is defined by comparison between the variance of estimation in the studied design Δ and that in a reference design Δ_r, ordinarily chosen as a completely randomized factorial design with one replication of each treatment. The model for this reference design Δ_r is $E(\mathbf{y}_r) = \boldsymbol{\tau} = X_r\boldsymbol{\alpha}$, where $\boldsymbol{\tau}$ is the vector of treatment effects. The vector $\boldsymbol{\alpha}$ will always be chosen so that $X_r^* X_r = |T|\mathbf{I}_p$. Hence the per unit information matrix D_r for $\boldsymbol{\alpha}_1$ in Δ_r is the identity matrix:

$$D_r = \frac{X_{r1}^* X_{r1}}{|T|} = \mathbf{I}_p \tag{4.8}$$

This is equivalent to saying that the coordinates α_i of $\boldsymbol{\alpha}$ are orthogonal linear forms of $\boldsymbol{\tau}$ of square norm $|T|$ [this is clearly satisfied by the vector $\boldsymbol{\alpha}$ in the model (2.9)].

The per unit information matrix D in Δ is

$$D = X_1^* Q_0 X_1 / n \tag{4.9}$$

The efficiency $\text{eff}(\mathbf{a})$ with respect to a real contrast $\mathbf{a}^*\boldsymbol{\alpha}_1$ ($\mathbf{a} \in \Theta_1$) is defined as 0 if the contrast is not estimable, as the ratio of variances $\mathbf{a}^* D_r^{-1}\mathbf{a}/\mathbf{a}^* D^- \mathbf{a}$ if it is estimable, – i.e. if $\mathbf{a}$ belongs to $\operatorname{Im} D$,

$$\text{eff}(\mathbf{a}) = \begin{cases} 0 & \text{if} \quad \mathbf{a} \notin \operatorname{Im} D \\ \mathbf{a}^* D_r^{-1}\mathbf{a}/\mathbf{a}^* D^- \mathbf{a} & \text{if} \quad \mathbf{a} \in \operatorname{Im} D \end{cases} \tag{4.10}$$

A good way to study the behaviour of the design with respect to the subspace E of real contrasts of $\boldsymbol{\alpha}_1$ is to examine the principal contrasts $\mathbf{a}_1^*\boldsymbol{\alpha}_1, \ldots, \mathbf{a}_q^*\boldsymbol{\alpha}_1$, where $\mathbf{a}_1$ is the vector in Θ_1 maximising $\text{eff}(\mathbf{a})$, $\mathbf{a}_2$ the vector in Θ_1 maximising $\text{eff}(\mathbf{a})$ with the constraint $\mathbf{a}D_r^{-1}\mathbf{a}_1 = 0$, and so on. The corresponding efficiencies will be called the principal efficiencies. The word "basic" has also been used instead of "principal" to define the similar notions in the context of block design [16].

By lemma 3.1, the mapping $\mathbf{b} \mapsto \mathbf{a} = N^*\mathbf{b}$ is a bijection from $\mathbb{R}_q$ onto Θ_1. To find the principal contrasts, we can therefore search for the vector $\mathbf{b}_1$ in $\mathbb{R}_q$ maximising $\text{eff}(N^*\mathbf{b})$, the vector $\mathbf{b}_2$ maximising the same quantity with the constraint $\mathbf{b}^*ND_r^{-1}N^*\mathbf{b}_1 = 0$, and so on. We then have: $\mathbf{a}_1 = N^*\mathbf{b}_1, \ldots, \mathbf{a}_q = N^*\mathbf{b}_q$ and the principal contrasts are $\mathbf{a}_1^*\boldsymbol{\alpha}_1 = \mathbf{b}_1'N\boldsymbol{\alpha}_1 = \mathbf{b}_1'\boldsymbol{\beta}_1, \ldots, \mathbf{a}_q^*\boldsymbol{\alpha}_1 = \mathbf{b}_q'\boldsymbol{\beta}_1$.

Let C and C_r be the per unit information matrices for $\boldsymbol{\beta}_1$ in Δ and Δ_r respectively:

$$C = N^{*-1}DN^{-1}, \quad C_r = N^{*-1}D_rN^{-1} \tag{4.11}$$

The replacement of $\mathbf{a}$ by $N^*\mathbf{b}$ in (4.10) gives

$$\text{eff}(N^*\mathbf{b}) = \begin{cases} 0 & \text{if } \ \mathbf{b} \notin \text{Im}\, C \\ \mathbf{b}^*C_r^{-1}\mathbf{b}/\mathbf{b}^*C^-\mathbf{b} & \text{if } \ \mathbf{b} \in \text{Im}\, C \end{cases} \tag{4.12}$$

The following proposition then follows from well known results of principal component analysis.

Proposition 4.4 *The principal efficiencies for the subspace E of real contrasts of $\boldsymbol{\alpha}_1$ are the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_q$ of the matrix CC_r^{-1}, which are equal to the eigenvalues of $DD_r^{-1} = X_1^*Q_0X_1/n$. If $\mathbf{b}_1, \ldots, \mathbf{b}_q$ are the corresponding eigenvectors of CC_r^{-1}, the principal contrasts are $\mathbf{b}_1'\boldsymbol{\beta}_1, \ldots, \mathbf{b}_q'\boldsymbol{\beta}_1$. They are equal to $\mathbf{a}_1^*\boldsymbol{\alpha}_1, \ldots, \mathbf{a}_q^*\boldsymbol{\alpha}_1$, where $\mathbf{a}_1 = N^*\mathbf{b}_1, \ldots, \mathbf{a}_q = N^*\mathbf{b}_q$ are D_r^{-1}-orthogonal eigenvectors of DD_r^{-1}.*

Hence studies of efficiency can be made directly on the complex information matrix provided the subset of parameters considered is stable by conjugation.

We shall now suppose that N is unitary, so that $C_r = \mathbf{I}_q$ and $CC_r^{-1} = C$. If a global mesure of efficiency $\text{eff}(E)$ for the contrasts in E is requested, a reasonable choice is

$$\text{eff}(E) = \frac{\psi(\mathbf{I}_q)}{/}psi(C) \tag{4.13}$$

where ψ satisfies conditions (a), (b) and (c) of the preceding section. By proposition 4.2, if the set J_1 of indices in $\boldsymbol{\alpha}_1$ satisfies (3.16) and if $\text{trace}(X_1^*X_1/(nq)) \leq 1$, then

$$\text{eff}(E) \leq 1 \tag{4.14}$$

The same inequality is in general not guaranteed for a single contrast, unless this contrast is in fact a real parameter α_j where $j = -j$, because we can then apply (4.14) to the one dimensional subspace E generated by α_j.

5 Cyclic designs

5.1 Introduction

Cyclic designs are block designs obtained by developing cyclically one or more initial blocks. A good account on them can be found in David [6]. Generalized cyclic designs were defined by John [8] as a generalization of cyclic designs using an arbitrary abelian group instead of a cyclic one. Bailey and Rowley (BR) [4] studied a still larger class of designs of the same kind, the construction of which involves a (non necessarily commutative) group of permutations of the set of treatments.

In order to avoid a multiplicity of notations, we shall group all these designs under the general denomination of *cyclic designs*. This is not a source of ambiguity since the context will always make clear the nature of the group involved and the precise type of construction. Moreover, the properties of the design do not really depend on the group being cyclic or commutative, so that there seem to be no reason other than historic to base the terminology on these characteristics of the group.

5.2 Operation of a group on a set

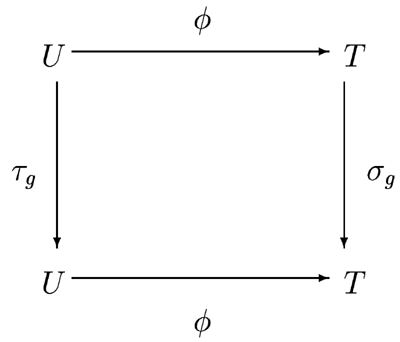
We let T be a set of treatments and G a (multiplicative) group operating on T (see [13]). The operation associates to each pair $(g, \mathbf{t})$ in $G \times T$ a product $g\mathbf{t}$ in T. Equipped with this product, T is called a *G-set*. To each element g in G is then associated a permutation $\sigma_g : \mathbf{t} \mapsto g\mathbf{t}$ of T, and the mapping $g \mapsto \sigma_g$ from G into the group S_T of permutations of T, is a group morphism.

We recall that the *orbit* of an element $\mathbf{t}$ in T is the set of all $g\mathbf{t}$ for $g \in G$. The different orbits form a partition of T. For a given $\mathbf{t} \in T$, the set of g in G such that $g\mathbf{t} = \mathbf{t}$ is a subgroup of G called the *stabilizer*, or *fixator*, or *isotropy group* of $\mathbf{t}$. It is generally denoted $G_{\mathbf{t}}$. There is a one to one mapping $gG_{\mathbf{t}} \mapsto g\mathbf{t}$ from the set $G/G_{\mathbf{t}}$ of left cosets of $G_{\mathbf{t}}$ onto the orbit of $\mathbf{t}$, and we shall often identify these two sets by this bijective mapping.

We also need the following classical definition:

Definition 5.1 (G-morphism) *Let U and T be G-sets. A G-morphism from U to T is a mapping ϕ satisfying $\phi(gx) = g\phi(x)$ for every $x \in U$ and $g \in G$.*

In other words, a G-morphism is a mapping making the following diagram, where τ_g and σ_g are the permutations induced by g on U and T respectively, commutative:



If F is a subgroup of G, there is a canonical operation of G on the set G/F of left cosets hF of F defined by:

$$g(hF) = (gh)F \ .$$

If H is a subgroup of G including F, the canonical surjection $gF \mapsto gH$ from G/F onto G/H is a G-morphism.

5.3 Definition and structure of cyclic designs

In the simplest case, a cyclic design is the set of blocks obtained by applying the permutations induced by the elements of G to an initial block. Such a set of blocks will be called a *cyclic set.* More generally, a *cyclic design* consists of a combination of several cyclic sets, possibly equal.

Following BR, we shall authorize the initial block to be a *multiset* of treatments. Roughly speaking, a multiset is a set $K = \{t_1, \dots, t_k\}$ which can contain the same element several times. To define it more precisely, we must specify the set $[K]$ of distinct elements and the number of occurrences of each of these elements. Hence a multiset K can be defined as a function K from $[K]$ into the set N of natural number: $K(\mathbf{t})$ is the number of times $\mathbf{t}$ appears in the multiset K. A multiset K can also be defined as a sequence $(t_1, \dots, t_k)$, provided sequences containing the same elements with the same numbers of occurrences are identified. We shall of course speak of multiset of treatments when the elements of the multiset are treatments. From the operation of G on T, an operation of G on multisets of treatments is derived:

$$g(t_1, \dots, t_k) = (gt_1, \dots, gt_k) \ .$$

With these notations, we have:

Definition 5.2 (Cyclic set) *Let K be a multiset with elements in the G-set T. The cyclic set with initial block K is the block design whose blocks are the multisets in the orbit of K.*

In BR terminology, this is called a thin (G, γ) design. γ refers to the partition into blocks of the set of experimental units.

In John [8] and David [6], the term "cyclic set" stands for the design having one block gK for every g, and the design given by definition 5.2. is called a "fractional" cyclic set. However, it is clear that the first design consists of repetition of the latter, and has therefore no special interest in itself. So, we prefer to use the more concise denomination "cyclic set" for the basic design given by definition 5.2.

Definition 5.3 (Cyclic design.) *Let $K_1, \ldots, K_d$ be multisets with elements in the G-set T and $B_1, \ldots, B_d$ be the sets of blocks of the cyclic sets having respectively $K_1, \ldots, K_d$ as initial blocks. Then, the cyclic design with initial blocks $K_1, \ldots, K_d$ is the block design having the disjoint union of $B_1, \ldots, B_d$ as set of blocks.*

In BR terminology, it is called a (G, γ) design. If $K_1, \ldots, K_d$ are equal, the cyclic design is said to be homogeneous. The cyclic sets defined in John and David are thus particular case of homogeneous cyclic designs.

Theorem 5.1 below gives a sufficient and necessary condition for a block design to be a cyclic design, in term of G-morphisms. A block design can be seen as a pair (ϕ_T, ϕ_B) of mappings from the set U of experimental units into the set T of treatments and the set B of blocks respectively : $\phi_T(\mathbf{u})$ and $\phi_B(\mathbf{u})$ are the treatment and block assigned to unit $\mathbf{u}$.

Theorem 5.1 *A block design (ϕ_T, ϕ_B) is a cyclic design with treatments in the G-set T if and only if it is possible to define operations of G respectively on U and B such that ϕ_T and ϕ_B are G-morphisms.*

This theorem is part of theorem 4.1 of BR, though expressed in different words. The following demonstration gives several interesting subresults which are not explicitely quoted in BR. We first give a description of a G-morphism as juxtaposition of quotient maps.

Definition 5.4 (Elementary G-morphism) *A G-morphism ϕ from A to B will be said to be elementary if G is transitive on A and on B simultaneously.*

Proposition 5.1 *Let G_a and G_b be subgroups of G such that G_a is included in a conjugate xG_bx^{-1} of G_b. Then the mapping $gG_a \mapsto gxG_b$ is an elementary G-morphism from G/G_a onto G/G_b. Conversely, any elementary G-morphism ϕ: $A \leftarrow B$ can be identified with such a mapping by taking G_a and G_b as the stabilizer of elements $a \in A$ and $b \in B$, and x such that $\phi(a) = xb$.*

Proof. Let G_a and G_b be subgroups of G such that $G_a \subset xG_bx^{-1}$. If g and h are in the same left coset of G_a, then $g^{-1}h \in G_a$, hence $x^{-1}g^{-1}hx \in x^{-1}G_ax \subset G_b$ and consequently gx and hx are in the same left coset of G_b. The mapping $gG_a \mapsto gxG_b$ is therefore well defined, and it is clearly an elementary G-morphism. Conversely, we have $\phi(ga) = gxb$. If we identify G/G_a with A and G/G_b with B by the mappings $gG_a \mapsto ga$ and $gG_b \mapsto gb$, the mapping ϕ is identified with the mapping $gG_a \mapsto gxG_b$ ■

Corollary 5.1 *Let ϕ: $A \longrightarrow B$ be an elementary G-morphism, a an element of A and b its image by ϕ: $b = \phi(a)$. Then the stabilizer G_a of a is included in the stabilizer G_b of b and ϕ can be identified with the canonical mapping $gG_a \mapsto gG_b$ from G/G_a onto G/G_b.*

If $A_1, \ldots, A_d$ are G-sets, we denote by $A_1 \sqcup \ldots \sqcup A_d$ the G-set which is the disjoint union of $A_1, \ldots, A_d$ with the operation naturally induced by that of $A_1, \ldots, A_d$. The next proposition states that any G-morphism can be built by juxtaposition of elementary G-morphisms.

Proposition 5.2 *Let $A_1, \ldots, A_d$, $B_1, \ldots, B_e$ be transitive G-sets and for $i = 1, \ldots, d$, let ϕ_i: $A_i \longrightarrow B_{f(i)}$ be an elementary G-morphism. Then the mapping ϕ from $A = A_1 \sqcup \ldots \sqcup A_d$ to $B = B_1 \sqcup \ldots \sqcup B_e$ which coincides with ϕ_i on A_i for $i = 1, \ldots, d$ is a G-morphism. Conversely, any G-morphism ϕ: $A \longrightarrow B$ can be obtained in this way. $A_1, \ldots, A_d$ are the orbits of G in A and $B_1, \ldots, B_e$ those of G in B. Then ϕ_i is the mapping coinciding with ϕ on A_i.*

The proof is straightforward.

Let now K be a multiset with elements in the G-set T, and G_K its stabilizer. The proposition below is the natural extension of a result of Dean and Lewis [7].

Proposition 5.3 *K is a disjoint union of orbits $G_K\mathbf{t}_1, \ldots, G_K\mathbf{t}_s$ for G_K, where $\mathbf{t}_1, \ldots, \mathbf{t}_s$ are s possibly equal treatments.*

Proof. If treatment $\mathbf{t}$ appears $K(\mathbf{t})$ times in K, every element $g\mathbf{t}$ in $G_K\mathbf{t}$ appears equally $K(\mathbf{t})$ times in $gK = K$. Hence, we can group the treatments in K as indicated in the proposition ■

Remark. If K is a disjoint union of sets $H\mathbf{t}$ for a subgroup H of G, then H stabilizes K and is thus included in G_K. This shows that G_K is maximal among these subgroups.

We consider now a cyclic design (ϕ_T, ϕ_B). We let K be the initial block of one of the constituent cyclic sets and B_K the corresponding set of blocks. If $g_1G_K, \ldots, g_bG_K$ are the distinct left cosets of G_K in G, these blocks are $g_1K, \ldots, g_bK$. By proposition 5.3, K is a disjoint union $G_K\mathbf{t}_1 \sqcup \ldots \sqcup G_K\mathbf{t}_s$. We can therefore write the cyclic set as in table 1 below. The corresponding set of units is partitioned into subsets $U_1, \ldots, U_s$ as indicated by the dotted lines. The treatments appearing in a given subset U_l are all the treatments in the orbit of $\mathbf{t}_l$. Consider then the $\mathbf{t}_l$ belonging to the orbit $T\mathbf{t}$ of a given element $\mathbf{t}$. Let $L(\mathbf{t})$ be the corresponding subset of indices l. For each $l \in L(\mathbf{t})$, choose x_l in G so that $\mathbf{t}_l = x_l\mathbf{t}$. Let G_K, G_t be the stabilizer of K and $\mathbf{t}$; $G_l = x_lG\mathbf{t}x_l^{-1}$ the stabilizer of $\mathbf{t}_l$; and finally $G_{Kl} = G_K \cap G_l$ the stabilizer of $\mathbf{t}_l$ in G_K. Then B_K and $T_\mathbf{t}$ can be identified with G/G_K and $G/G_\mathbf{t}$ by the bijections $gK \mapsto gG_K$ and $gt \mapsto gG_\mathbf{t}$ respectively. We then have:

Proposition 5.4 *If $l \in L(\mathbf{t})$, we can identify U_l with G/G_{Kl} and the mappings ϕ_B and ϕ_T restricted to U_l with the elementary morphisms $gG_{Kl} \mapsto gG_K$ and $gG_{Kl} \mapsto gx_lG_\mathbf{t}$.*

block g_1K	⋮	$g_1G_Kt_1$	⋮		⋮	$g_1G_Kt_s$	⋮
block g_2K	⋮	$g_2G_Kt_1$	⋮		⋮	$g_2G_Kt_s$	⋮
⋮	⋮	⋮	⋮	…	⋮	⋮	⋮
block;g_bK	⋮	$g_bG_Kt_1$	⋮		⋮	$g_bG_Kt_s$	⋮
	⋮	$\underbrace{\qquad}_{U_1}$				$\underbrace{\qquad}_{U_s}$	

Table 1:

Proof. Let $h_1G_{Kl}, \ldots, h_rG_{Kl}$ be the elements of G_K/G_{Kl}. The elements of the orbit $G_K\mathbf{t}_l$ are $h_1\mathbf{t}_l, \ldots, h_r\mathbf{t}_l$ and can be identified with the left cosets $h_1G_{Kl}, \ldots, h_rG_{Kl}$. The set of treatments $g_jG_K\mathbf{t}_l$ appearing in the block g_jK and subset of units U_l is equal to $\{g_jh_1\mathbf{t}_l, \ldots, g_jh_r\mathbf{t}_l\}$ and can be identified with the left cosets $g_jh_1G_{Kl}, \ldots, g_jh_rG_{Kl}$ of G_{Kl} in g_jG_K. The whole subset U_l can therefore be identified with the set G/G_{Kl} of left cosets of G_{Kl} in G, the elements of which are precisely the left cosets $g_jh_iG_{Kl}$, where i varies from 1 to r and j from 1 to b. If we make this identification, the treatment assigned to unit $g_jh_iG_{Kl}$ is $g_jh_i\mathbf{t}_l = g_jh_ix_l\mathbf{t}$ whereas the corresponding block is $g_jh_iK = g_jK$ ■

The "only if" part of theorem 5.1 is a consequence of this proposition and of proposition 5.2. Indeed, the disjoint union $U_{K\mathbf{t}} = \sqcup_{l\in L(\mathbf{t})}U_l$ contains all experimental units in B_K with treatments in $T_\mathbf{t}$ and is therefore equal to $U_{K\mathbf{t}} = \phi_B^{-1}(B_K) \cap \phi_T^{-1}(T\mathbf{t})$. The set U is itself the disjoint union of the subsets $U_{K\mathbf{t}}$, where K describes the initial blocks and $T_\mathbf{t}$ the orbits of G in T.

The "if" part of the theorem follows from the next proposition.

Proposition 5.5 *Suppose, in a given block design, that the mappings giving respectively the treatment and block assigned to each unit are two G-morphisms $\phi_T : U \mapsto T$ and $\phi_B : U \mapsto B$. Then, the reciprocal image of the elements of a given orbit in B constitute an homogeneous cyclic design. The initial block is the multiset K of treatments applied to the set of units $\phi_B^{-1}(\mathbf{b})$, where $\mathbf{b}$ is a representative of this orbit.*

Proof. If V is a subset of U, we shall denote by $K(V)$ the multiset of treatments assigned by ϕ_T to the units of V. Since ϕ_T is a G-morphism, we have $K(gV) = gK(V)$. Note that $K = K\left(\phi_B^{-1}(\mathbf{b})\right)$.

For every g in G, we have $\phi_B^{-1}(g\mathbf{b}) = g\phi_B^{-1}(\mathbf{b})$. Hence, for every g in the stabilizer $G_\mathbf{b}$ of $\mathbf{b}$ in G, we have $\phi_B^{-1}(\mathbf{b}) = \phi_B^{-1}(g\mathbf{b}) = g\phi_B^{-1}(\mathbf{b})$ and consequently $K = gK$. It follows that the stabilizer $G_\mathbf{b}$ of $\mathbf{b}$ is included in the stabilizer G_K of the multiset K. Let $h_1G_\mathbf{b}, \ldots, h_rG_\mathbf{b}$ be the left cosets of $G_\mathbf{b}$ in G_K and $g_1G_K, \ldots, g_sG_K$ those of G_K in G. The elements of $G/G_\mathbf{b}$ are the $r \times s$ left cosets $g_jh_iG_\mathbf{b}$, and the blocks in the orbit of $\mathbf{b}$ are therefore the $r \times s$ blocks $g_jh_i\mathbf{b}$. The multiset in block $g_jh_i\mathbf{b}$ is $K\left[\phi_B^{-1}(g_jh_i\mathbf{b})\right] = K\left[g_jh_i\phi_B^{-1}(\mathbf{b})\right] = g_jh_iK = g_jK$. Hence, each of the multisets g_jK of the cyclic set with initial block K appears in exactly r blocks: the blocks $g_j h_1 \mathbf{b}, \ldots, g_j h_r \mathbf{b}$ associated to the left cosets of $G_{\mathbf{b}}$ in $g_j G_K$, q.e.d. ■

6 The linear model of a cyclic design

6.1 Homogeneous decomposition

We consider a cyclic design defined by two G-morphisms $\phi_T : U \mapsto T$ and $\phi_B : U \mapsto B$. To simplify the description of the associated linear model, we shall use a classical result of the theory of linear representations known as Schur's lemma. We first briefly give some notations and results of this theory. A more detailed account can be found in BR [4], Serre [17], Ledermann [14], or in Lang [13] (chap XVII, XVIII).

If T is a G-set, each $g \in G$ induces a permutation σ_g on T and a linear endomorphism R_g of $\mathbb{C}^T$ defined by

$$R_g(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \circ \sigma_g^{-1} \tag{6.1}$$

The inverse σ_g^{-1} of σ_g is also the permutation associated to g^{-1}. Hence we have $\sigma_g^{-1}(\mathbf{t}) = g^{-1}\mathbf{t}$ for $\mathbf{t} \in T$ and the coordinate of $R_g(\boldsymbol{\alpha})$ on $\mathbf{t}$ is

$$R_g(\boldsymbol{\alpha})(\mathbf{t}) = \boldsymbol{\alpha} \circ \sigma_g^{-1}(\mathbf{t}) = \boldsymbol{\alpha}\left(g^{-1}\mathbf{t}\right), \tag{6.2}$$

where $\boldsymbol{\alpha}\left(g^{-1}\mathbf{t}\right)$ is the coordinate of $\boldsymbol{\alpha}$ on $g^{-1}\mathbf{t}$ (recall that $\mathbb{C}^T$ can be considered as the set of mappings from T to $\mathbb{C}$ so that the composite map of $\sigma_g^{-1} : T \to T$ and $\boldsymbol{\alpha} : T \to C$ is defined and belongs to $\mathbb{C}^T$).

If $(\mathbf{e_t})_{\mathbf{t} \in T}$ is the canonical basis of $\mathbb{C}^T$, we immediately deduce from (6.2) that

$$R_g(\mathbf{e_t}) = \mathbf{e}_{g\mathbf{t}} \,. \tag{6.3}$$

The column $\mathbf{t}$ in the matrix of $\mathbb{R}_g$ therefore has 1 in the row $g\mathbf{t}$ and 0 elsewhere. This matrix is called the *permutation matrix* of σ_g and will also be denoted by R_g if there is no risk of confusion.

The mapping $g \mapsto R_g$ is a linear representation of G in $\mathbb{C}^T$, called the permutation representation of G in $\mathbb{C}^T$. This representation makes $\mathbb{C}^T$ into a G-module (Serre [17], chap 6), the product $g\boldsymbol{\alpha}$ of $g \in G$ by $\boldsymbol{\alpha} \in \mathbb{C}^T$ being defined by

$$g\boldsymbol{\alpha} = R_g(\boldsymbol{\alpha}) \tag{6.4}$$

We now recall some important results about G-modules. A *submodule* W of a G-module E is a subspace of E which is G-invariant, that is to say satisfies $gW \subset W$ for every $g \in G$. A G-module different from $\{0\}$ is irreducible (one also says simple) if it admits no proper submodule.

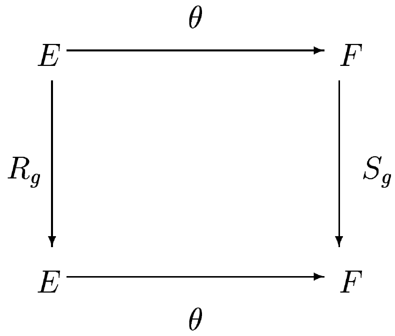
Theorem 6.1 (Maschke) . *Every G-module E admits a direct sum decomposition $E = W_1 \oplus \ldots \oplus W_k$ into irreducible submodules $W_1, \ldots, W_k$.*

Such a decomposition will be called an *irreducible decomposition.*

If E and F are G-modules, a G-homorphism $\theta : E \to F$ is a linear mapping satisfying

$$\forall \boldsymbol{\alpha} \in E, \forall g \in G, \quad \theta(g\boldsymbol{\alpha}) = g\theta(\boldsymbol{\alpha}) \tag{6.5}$$

Alternatively, it can be defined as a linear mapping making the following diagram commutative for every $g \in G$:



Here, R_g and S_g are the linear mappings induced by g on E and F respectively.

The fundamental result concerning G-homomorphism is:

Theorem 6.2 (Schur's lemma) *Let E and F be irreducible G-modules. Then a G-homomorphism $\theta : E \to F$ is either an isomorphism, or else the zero map.*

If $\theta : E \to F$ is a G-homomorphism, Schur's lemma can be applied to the components of the irreducible decompositions of E and F to get a simplified description of θ. To be more precise, we first introduce some notations.

Let $V_1, \ldots, V_r$ be the distinct non-isomorphic irreducible G-modules, and $\chi_1, \ldots, \chi_r$ the associated *(irreducible) characters* [14, 17]. Consider then, in an irreducible decomposition $E = W_1 \oplus \ldots \oplus W_k$, the sum E_i of submodules W_j isomorphic to V_i. This submodule E_i is in fact the sum of all submodules of E isomorphic to V_i and is therefore independant of the irreducible decomposition considered. If there are n_i submodules isomorphic to V_i in the irreducible decomposition, E_i is G-isomorphic to the n_i-fold direct sum of V_i, and has $n_i\chi_i$ as character. It will be called the G-homogeneous subspace associated to χ_i. Note that E_i can be reduced to $\{0\}$.

E can be decomposed into the direct sum of its homogeneous subspace:

$$E = E_1 \oplus \ldots \oplus E_r \tag{6.6}$$

This decomposition, coarser than any irreducible one, is called the *G-homogeneous decomposition* of E (BR). The character π of E is equal to $\sum n_i\chi_i$, and n_i is called the *multiplicity* of χ_i in π. If $n_i = 0$ or 1 for every i, π is said to be *multiplicity free.*

The following corollary of Schur's lemma shows how the study of a general G-homomorphism can be reduced to that of G-homomorphisms between two direct sums of the same irreducible G-module.

Corollary 6.1 *Let $\theta : E \to F$ be a G-homorphism and for each irreducible character χ_i of G, let E_i, F_i be the homogeneous subspace of E and F respectively associated to χ_i. Then we have $\theta(E_i) \subset F_i$.*

The G-homomorphism θ can thus be decomposed into a block diagonal form $\theta = \text{diag}(\theta_1, \dots, \theta_r)$ where θ_i is the G-homomorphism from E_i to F_i coinciding with θ on E_i.

We now go back to the case of a permutation representation $g \mapsto R_g$ of G in $\mathbb{C}^T$. The mappings R_g act by permutation of the coordinates (see (6.2)), hence they are unitary for the usual scalar product of $\mathbb{C}^T$:

$$\langle R_g \boldsymbol{\alpha}, R_g \boldsymbol{\beta} \rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \ . \tag{6.7}$$

It follows (Serre, section 1.3) that the W_i in the irreducible decomposition of $\mathbb{C}^T$, can be chosen orthogonal, so that:

Proposition 6.1 *Let T be a G-set and $\mathbb{C}^T$ the corresponding G-module. The homogeneous decomposition of $\mathbb{C}^T$ is orthogonal with respect to the usual scalar product.*

The following results link this section to section 4 on the complex model.

Proposition 6.2 *Let $g \mapsto R_g$ be a permutation representation of G in $\mathbb{C}^T$ and W an irreducible submodule of $\mathbb{C}^T$, of character χ. Then $\overline{W}$ is also an irreducible submodule of $\mathbb{C}^T$, of character $\overline{\chi}$, which is either equal to W, or else has an intersection with W reduced to $\{0\}$.*

Proof. If Ω is a G-invariant subspace, $R_g(\Omega) \subset \Omega$ for each $g \in G$. Then since the matrix R_g is real, $R_g(\overline{\Omega}) \subset \overline{\Omega}$ for each $g \in G$, and $\overline{\Omega}$ is also G-invariant. W being G-invariant, $\overline{W}$ is also G-invariant. If Ω were a proper submodule of $\overline{W}$, $\overline{\Omega}$ would be a proper submodule of W, which is absurd. Hence $\overline{W}$ is also irreducible. The intersection $W \cap \overline{W}$ is a G-invariant subspace of W, hence it is $\{0\}$ or W. If it is W, we have $W \subset \overline{W}$, hence $W = \overline{W}$ because $\overline{W}$ is irreducible.

Finally, if $(\mathbf{a}_i)$ is a basis of W and (λ_{ij}) the matrix of R_g on W with this basis, we have $R_g \mathbf{a}_j = \sum \lambda_{ij} \mathbf{a}_i$, hence $R_g \overline{a}_j = \sum \overline{\lambda_{ij}} \overline{\mathbf{a}}_i$. The matrix of R_g on $\overline{W}$, with the basis $(\overline{\mathbf{a}}_i)$,is therefore $(\overline{\lambda_{ij}})$. Since trace $\left(\overline{\lambda_{ij}}\right)$ is the conjugate of trace (λ_{ij}), the value on g of the character of $\overline{W}$ is $\overline{\chi}(g)$ and the character of $\overline{W}$ is $\overline{\chi}$. ■

It is easy, using proposition 6.2, to modify the recurrent process leading to an irreducible decomposition $\mathbb{C}^T = W_1 \oplus \dots \oplus W_k$ so that the W_i are either self-conjugate, or conjugate in pairs.

Consider now the homogeneous decomposition $\mathbb{C}^T = E_1 \oplus \dots \oplus E_r$, where E_i is the sum of the irreducible submodules W of character χ_i. $\overline{E_i}$ is the sum of the irreducible submodules $\overline{W}$ of character $\overline{\chi}_i$. If χ_i is real $(\chi_i = \overline{\chi}_i)$, it is equal to E_i. Otherwise, it is equal to the homogeneous subspace E_j of character $\chi_j = \overline{\chi}_i$, which is distinct from E_i. Hence we have:

Proposition 6.3 *Let $g \mapsto R_g$ be a permutation representation of G in $\mathbb{C}^T$. Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G, and for $i = 1, \ldots, r$ let E_i the homogeneous (possibly null) subspace associated to χ_i. Then $\mathbb{C}^T$ is the direct orthogonal sum of the E_i. If χ_i is real, E_i is self-conjugate. If χ_i and χ_j are distinct conjugate characters, E_i and E_j are conjugate. The E_i can be further decomposed into direct sum of irreducible submodules W_{ij}: $E_i = W_{i1} \oplus \ldots \oplus W_{in}$, in such a way that (i)if χ_i is real, the W_{ij} are either self-conjugate or conjugate in pairs, (ii)if χ_i and χ_j are distinct conjugate characters, $n_i = n_j$ and $W_{jl} = \overline{W}_{il}$.*

Let now $E = V_1 \oplus \ldots \oplus V_m$ be a decomposition of E such that the V_i are either self-conjugate or conjugate in pairs. We can choose a basis $(\mathbf{e}_{ij})$ such that, for a fixed i, $(\mathbf{e}_{ij})_j$ is a basis of V_i and:

(1) if $V_i = \overline{V}_i$, the $\mathbf{e}_{ij}$ are real (proposition 3.1)

(2) if $V_j = \overline{V}_i$ and $j \neq i$, then $\mathbf{e}_{jl} = \overline{\mathbf{e}}_{il}$.

We shall say that such a basis is coherent with the decomposition $E = V_1 \oplus \ldots \oplus V_m$.

To use the corollary of theorem 6.2 (Shur's lemma) in the context of cyclic designs, we need a supplementary result. For each mapping $\phi_T : U \to T$, we let $X_T : \mathbb{C}^T \to \mathbb{C}^U$ be the linear mapping induced by ϕ_T, which is defined by $X_T(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \circ \phi_T$.

Proposition 6.4 *If $\phi_T : U \to T$ is a G-morphism, $X_T : \mathbb{C}^T \to \mathbb{C}^U$ is a G-homomorphism.*

Proof. Let ρ_g and σ_g be the permutations induced by $g \in G$ on T and U respectively, and R_g and S_g the corresponding linear mapping of $\mathbb{C}^T$ and $\mathbb{C}^U$. By definition of R_g, S_g and X_T, we have:

$$R_g(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \circ \rho_g^{-1}, \quad S_g(\boldsymbol{\beta}) = \boldsymbol{\beta} \circ \sigma_g^{-1}, \quad X_T(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \circ \phi_T \ .$$

Since ϕ_T is a G-morphism, we have for every $g \in G$: $\rho_g^{-1} \circ \phi_T = \phi_T \circ \sigma_g^{-1}$. For every $\boldsymbol{\alpha} \in \mathbb{C}^T$, we then have $\boldsymbol{\alpha} \circ \phi_T \circ \sigma_g^{-1} = \boldsymbol{\alpha} \circ \rho_g^{-1} \circ \phi_T$, which is equivalent to

$$S_g(X_T(\boldsymbol{\alpha})) = X_T(R_g(\boldsymbol{\alpha})) \ .$$

Hence $S_g X_T = X_T R_g$ for every g and X_T is a G-homomorphism ■

Let now Δ be the cyclic design defined by the G-morphisms $\phi_T : U \to T$ and $\phi_B : U \to B$, where U is the G-set of experimental units, T the G-set of treatments and B the G-set of blocks. Let $X_T : \mathbb{C}_T \to \mathbb{C}^U$ and $X_B : \mathbb{C}^B \to \mathbb{C}^U$ be the G-homorphisms induced by ϕ_T and ϕ_B respectively. If the corresponding matrices in the canonical basis are also denoted by X_T and X_B, the model associated to Δ is

$$E(\mathbf{y}) = X_T \boldsymbol{\tau} + X_B \boldsymbol{\xi} \ , \tag{6.8}$$

where $\boldsymbol{\tau} \in \mathbb{R}_T$ is the vector of treatment effects and $\boldsymbol{\xi}$ the vector of block effects. A useful reparametrization can be obtained by decomposing $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$ on the homogeneous decompositions $\mathbb{C}^T = E_{T1} \oplus \ldots \oplus E_{Tr}$ and $\mathbb{C}^B = E_{B1} \oplus \ldots \oplus E_{Br}$. In order to obtain a model of the form (3.1) which satifies (3.2) and (3.3), matrices $A_T = (A_{T1}, \ldots, A_{Tr})$ and $A_B = (A_{B1}, \ldots, A_{Br})$ are chosen such that:

(i) their columns are basis coherent with the decompositions $\mathbb{C}^T = E_{T1} \oplus \ldots \oplus E_{Tr}$ and $\mathbb{C}^B = E_{B1} \oplus \ldots \oplus E_{Br}$ respectively.

(ii) the columns of A_{Ti} and A_{Bi} span E_{Ti} and E_{Bi} respectively.

We then have

$$\boldsymbol{\tau} = A_T \boldsymbol{\alpha}_T = A_{T1}\boldsymbol{\alpha}_{T1} + \cdots + A_{Tr}\boldsymbol{\alpha}_{Tr} \ , \tag{6.9}$$

$$\boldsymbol{\xi} = A_B \boldsymbol{\alpha}_B = A_{B1}\boldsymbol{\alpha}_{B1} + \cdots + A_{Br}\boldsymbol{\alpha}_{Br} \ , \tag{6.10}$$

and the model (6.8) can be written in the two following forms, where $X = (X_T A_T, X_B A_B)$ and $X_i = (X_T A_{Ti}, X_B A_{Bi})$:

$$E(\mathbf{y}) = X\boldsymbol{\alpha} = X_T A_T \boldsymbol{\alpha}_T + X_B A_B \boldsymbol{\alpha}_B (6.11) \tag{6.11}$$

$$\begin{aligned} E(\mathbf{y}) &= X_1\boldsymbol{\alpha}_1 + \cdots + X_r\boldsymbol{\alpha}_r \\ &= X_T A_{T1}\boldsymbol{\alpha}_{T1} + X_B A_{B1}\boldsymbol{\alpha}_{B1} + \cdots + X_T A_{Tr}\boldsymbol{\alpha}_{Tr} + X_B A_{Br}\boldsymbol{\alpha}_{Tr} \ . \end{aligned} \tag{6.12}$$

If $E_{U1} \oplus \ldots \oplus E_{Ur}$ is the homogeneous decomposition of $\mathbb{C}^U$, the G-homomorphism X_T sends $E_{Ti} = \operatorname{Im} A_{Ti}$ into E_{Ui}, and similarly X_B sends $E_{Bi} = \operatorname{Im} A_{Bi}$ into E_{Ui} (by the corollary of theorem 6.2). Since the homogeneous subspaces E_{Ui} are orthogonal, the blocks X_i are orthogonal to each other. It follows that the matrices $Q_B X_T A_{Ti}$, where $Q_B = \mathbf{I} - X_B(X_B^* X_B)^- X_B^*$ is the operator of orthogonal projection onto the orthogonal supplementary of $\operatorname{Im} X_B$, are orthogonal. Consequently, the per unit information matrix D for the vector $\boldsymbol{\alpha}_T$ of treatment parameters can be put under a block diagonal form:

$$D = A_T^* X_T^* Q_B X_T A_T / n = \operatorname{diag}(D_1, \cdots, D_r) \tag{6.13}$$

where D_i is the per unit information matrix for $\boldsymbol{\alpha}_{Ti}$, equal to

$$D_i = A_{Ti}^* X_T^* Q_B X_T A_{Ti} / n \ . \tag{6.14}$$

By construction, the matrices A_{Ti} are either real if χ_i is real, or conjugate by pairs ($A_{Tj} = \overline{A}_{Ti}$ if $\chi_j = \overline{\chi}_i$. Since $X_T^* Q_B X_T$ is real, the same property holds for the matrices D_i: D_i is real if χ_i is real and D_j and D_i are conjugate if χ_j and χ_i are conjugate complex irreducible characters.

We index the columns of A_T (and the coordinates of $\boldsymbol{\alpha}_T$) by a set J_T, and define the opposite $-j$ of each $j \in J_T$ as follows: (1) if j indexes a column of an A_{Ti} such that χ_i is real, $-j = j$;

(2) if j indexes a column of an A_{Ti} such that χ_i is not real, $-j$ is the index of the conjugate column in the matrix $A_{Tj} = \overline{A}_{Ti}$.

We can also index the columns of A_B and coordinates of $\boldsymbol{\alpha}_B$ by a set J_B and define the opposite $-j$ on J_B in a similar way. If J is the disjoint union of J_T and J_B, the matrix X of model (6.11) satisfies the condition (3.2) and $\boldsymbol{\alpha}$ consequently satisfies (3.3) (by Lemma 3.1). All the requirements for a complex linear model are therefore satisfied by the model (6.11) and the development of section 3 applies. Since we are mainly interested in treatment contrasts, we define Θ_T as the set of vectors $\boldsymbol{\alpha}_T = (\alpha_j)$, $j \in J_T$, whose coordinates associated to opposite elements of J_T are conjugate:

$$\Theta_T = \{\boldsymbol{\alpha}_T | \forall j \in J_T,\ \alpha_{-j} = \overline{\alpha}_j\} \tag{6.15}$$

Besides leading to a block diagonal information matrix, the decomposition (6.9) of $\boldsymbol{\tau}$ on the homogeneous subspaces often has another advantage. In several circumstances, it bears a sensible relationship with the decomposition of interest for the experimenter. This is illustrated by the reparametrization (2.5) which is a particular case of (6.9) (see below). Another interesting example using a non abelian group is given by BR (example 5.3).

The Homogeneous Subspaces in the Commutative Case. Suppose that T and G are abelian additive groups and that $\phi : G \to T$ is a group morphism. Let G operate on T by $g\mathbf{t} = \mathbf{t} + \phi(g)$. Once cyclic group decompositions of G and T have been chosen, ϕ is identified with its matrix and its dual $\phi^\times$ can be defined. Then the homogeneous subspace associated to the irreducible character $\boldsymbol{\eta}^{g^\times}$ is the subspace spanned by the vectors $\boldsymbol{\eta}^{\mathbf{t}^\times}$ where $\mathbf{t}^\times$ satisfies $\phi^\times \mathbf{t}^\times = -g^\times$. Indeed, if R_g is the permutation representation of g in $\mathbb{C}^T$ and $\phi^\times \mathbf{t}^\times = -g^\times$, we have

$$\begin{aligned} R_g\left(\boldsymbol{\eta}^{\mathbf{t}^\times}\right)(\mathbf{t}) &= \boldsymbol{\eta}^{\mathbf{t}^\times}(\mathbf{t} - \phi g) = \eta^{[\mathbf{t}^\times, \mathbf{t} - \phi g]} \\ &= \eta^{[-\phi^\times \mathbf{t}^\times, g]} \eta^{[\mathbf{t}^\times, \mathbf{t}]} = \boldsymbol{\eta}^{g^\times}(g) \boldsymbol{\eta}^{\mathbf{t}^\times}(\mathbf{t}) \end{aligned}$$

Thus $R_g\left(\boldsymbol{\eta}^{\mathbf{t}^\times}\right) = \boldsymbol{\eta}^{g^\times}(g)\boldsymbol{\eta}^{\mathbf{t}^\times}$, and the subspace spanned by $\boldsymbol{\eta}^{\mathbf{t}^\times}$ is irreducible of character $\boldsymbol{\eta}^{g^\times}$, q.e.d.

If G is an abelian additive group operating on T, each orbit $T_\mathbf{s}$ of an element $\mathbf{s}$ in T is G-isomorphic to $G/G_\mathbf{s}$. Recall that the operation of $G/G_\mathbf{s}$ is defined, for $g \in G$ and $\mathbf{t} \in G/G_\mathbf{s}$, by $g\mathbf{t} = \mathbf{t} + \phi_\mathbf{s}(g)$, where $\phi_\mathbf{s} : G \to G/G_\mathbf{s}$ is the canonical surjection. The above result can thus be applied to each orbit and, by imbedding the associated characters $\boldsymbol{\eta}^{\mathbf{t}^\times}$ into $\mathbb{C}^T$, an explicit irreducible decomposition of $\mathbb{C}^T$ is obtained.

If in a cyclic design built with a G-set of treatments, G is commutative, the above consideration can be used in conjunction with proposition 5.4 to get the information matrices D_i explicitly. Note that if G operates transitively on T, the homogeneous subspaces E_{Ti} are one dimensional. Now if the irreducible decomposition of $\mathbb{C}^B$ is of the type mentioned above, the D_i are of the form:

$$D_i = \begin{bmatrix} a & \mathbf{b}^* \\ \mathbf{b} & \Lambda \end{bmatrix},$$

where a is a scalar associated with the treatment contrast in E_{Ti}, Λ a diagonal matrix of size the number of cyclic sets, and b a column vector. The detailed calculation of D_i

is not given but the reader can find in section 7 a similar type of calculation completely developed. The efficiency for the treatment contrast can be deduced from D_i. The result is the same as that obtained in a slightly more general case in section 6.3.

6.2 Efficiency for treatment contrasts

Suppose that the homogeneous subspaces of the G-module $\mathbb{C}^T$ are irreducible, or equivalently that the characters of $\mathbb{C}^T$ are multiplicity free. Under this hypothesis, a cyclic design with elements in the G-set T has a property of general balance which implies that the efficiency is the same for all the contrasts belonging to a homogeneous subspace of $\mathbb{R}^T$ (see BR [4] theorem 5.6). This efficiency was given by BR in the case of abelian group design. We shall show that their formula also hold in the non abelian case, and shall give for the abelian case an interesting alternative form of it.

We consider again the cyclic design Δ defined by the two G-morphisms ϕ_T and ϕ_B. The model for the reference design Δ_r (see section 4.2) is $E(\mathbf{y}_r) = A_T\boldsymbol{\alpha}_T$. The corresponding per unit information matrix is $D_r = A_T^* A_T/|T|$. To simplify the expression of the efficiency, the columns of the matrices A_{Ti} are chosen orthogonal and of square norm $|T|$, so that

$$D_r = \frac{A_T^* A_T}{|T|} = \mathbf{I}_{|T|} \tag{6.16}$$

To obtain principal contrasts and efficiencies, we are then led to search for the eigenspaces and eigenvalues of the per unit information matrix D appearing in (6.13). Since D is block diagonal, this search can be done separately for each D_i. To each eigenvector $\mathbf{a}$ of D_i is associated an eigenvector $\mathbf{a}_T$ of D having the same eigenvalue λ as $\mathbf{a}$. Written in partitioned form, $\mathbf{a}_T$ has $\mathbf{a}$ in the ith position and 0 elsewhere: $\mathbf{a}_T = (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{a}', \mathbf{0}, \ldots, \mathbf{0})'$. The corresponding linear form of $\boldsymbol{\alpha}_T$ is $\mathbf{a}^*\boldsymbol{\alpha}_{Ti} = \mathbf{a}_T^*\boldsymbol{\alpha}_T = a_T^* A_T^* \boldsymbol{\tau}/|T|$. Notice that $A_T\mathbf{a}_T = A_{Ti}\mathbf{a}$ is then an eigenvector of $C = |T|X_T^* Q_B X_T/n$ having the same eigenvalue λ and belonging to the homogeneous subspace E_{Ti}. If χ_i is real, D_i is real and $\mathbf{a}$ can be chosen real. $\mathbf{a}^*\boldsymbol{\alpha}_{Ti}$ is then real and defines a principal contrast. If χ_i is not real, and $\chi_j = \overline{\chi}_i$, then $\overline{\mathbf{a}}$ is an eigenvector of D_j with the same eigenvalue λ. The vectors $\mathbf{a}_T$ and $\overline{\mathbf{a}}_T$ can be combined to form two new orthogonal eigenvectors of D, with the same eigenvalue λ, and belonging to Θ_T: $(\mathbf{a}_T + \overline{\mathbf{a}}_T)$, $(\mathrm{i}\mathbf{a}_T - \mathrm{i}\overline{\mathbf{a}}_T)$. These two vectors define two principal contrasts having the same principal efficiency λ.

To determine explicitly the principal efficiencies, we need the following results:

Proposition 6.5 *Let $g \mapsto R_g$ be a representation of G in a complex vector space F and $E_1 \oplus \ldots \oplus E_r$ be the homogeneous decomposition of F. The projector P_i of F onto E_i associated to this decomposition is given by the following formula, where χ_i is the irreducible character associated to E_i:*

$$P_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \overline{\chi}_i(g) R_g \ .$$

Proof. See Serre [17], theorem 8)

P_i will be called the canonical projector on E_i

Proposition 6.6 *Let $\Phi : F \to H$ be a G-homomorphism and P_F and P_H be the canonical projectors onto the homogeneous subspaces associated to a given irreducible character χ in F and H respectively. Then, we have $\Phi P_F = P_H \Phi$.*

Proof. Let $g \mapsto R_g$ and $g \mapsto S_g$ be the representation of G in F and H respectively. We have

$$P_F = \frac{\chi(1)}{|G|} \sum_{‘\in G} \overline{\chi}(g) R_g, \quad P_H = \frac{\chi(1)}{|G|} \sum_{‘\in G} \overline{\chi}(g) S_g,$$

whence:

$$\Phi P_F = \frac{\chi(1)}{|G|} \sum_{g\in G} \overline{\chi}(g) \Phi R_g = \frac{\chi(1)}{|G|} \sum_{g\in G} \overline{\chi}(g) S_g \Phi == P_H \Phi \blacksquare$$

Proposition 6.7 *Let $g \mapsto R_g$ be a permutation representation of G in $\mathbb{C}^T$ and Q the operator of orthogonal projection onto a G-invariant subspace of $\mathbb{C}^T$. Then Q commutes with the R_g ($QR_g = R_g Q$), i.e. Q is a G-homomorphism.*

Proof. Let W_1 be this G-invariant subspace and W_2 its orthogonal complement. Since the R_g are unitary, W_2 is also G-invariant. If $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ is the decomposition of a vector $\boldsymbol{\alpha}$ of $\mathbb{C}^T$ on W_1 and W_2, the decomposition of $R_g\boldsymbol{\alpha}$ on W_1 and W_2 is $R_g\boldsymbol{\alpha} = R_g\boldsymbol{\alpha}_1 + R_g\boldsymbol{\alpha}_2$. We then have $Q\boldsymbol{\alpha} = \boldsymbol{\alpha}_1$, $QR_g\boldsymbol{\alpha} = R_g\boldsymbol{\alpha}_1$, and $QR_g\boldsymbol{\alpha} = R_g\boldsymbol{\alpha}_1 = R_g Q\boldsymbol{\alpha}$ $\blacksquare$

We now go back to the cyclic design defined by the G-morphisms $\phi_T : U \to T$ and $\phi_B : U \to B$. We denote by R_g and S_g the permutation matrices (and corresponding operators) of g in ^T_C and $\mathbb{C}^U$ respectively. Since the linear mapping $X_T : \mathbb{C}^T \to \mathbb{C}^U$ induced by ϕ_T is a G-homomorphism, we have for every $g \in G$

$$X_T R_g = S_g X_T \tag{6.17}$$

This implies $R_g^* X_T^* = X_T^* S_g^*$. Multiplying this last equality by R_g on the left, S_g on the right, we obtain

$$X_T^* S_g = R_g X_T^* \tag{6.18}$$

It is clear that $\operatorname{Im} X_B$ is a G-invariant subspace of $\mathbb{C}^U$. Hence by proposition 6.7, Q_B commutes with S_g:

$$Q_B S_g = S_g Q_B \tag{6.19}$$

From the above equalities, we immediately deduce:

Proposition 6.8 *$X_T^* Q_B X_T$ commutes with R_g for every g.*

When all the blocks have the same size, this result also follows from theorem 4.2 of BR [4]. As a consequence of it, we have:

Proposition 6.9 *If the homogeneous subspace E_{Ti} of $\mathbb{C}^T$ is irreducible, it is included in an eigenspace of $X_T^* Q_B X_T$.*

Proof. See BR [4], theorem 5.1 (with $\mathbb{C}$ instead of $\mathbb{R}$) and remarks following it.

As a matter of fact, theorem 5.1 of BR [4] shows that an irreducible decomposition $\oplus_i W_i$ of $\mathbb{C}^T$ can be found such that each W_i is included in an eigenspace of $X_T^* Q_B X_T$. The number of distinct eigenvalues of $X_T^* Q_B X_T$ on E_{Ti} is thus at most the multiplicity of χ_i in the character of the permutation representation.

If E_{Ti} is an homogeneous irreducible subspace of $\mathbb{C}^T$ and ζ the eigenvalue of $X_T^* Q_B X_T$ on it, we have $X_T^* Q_B X_T A_{Ti} = \zeta A_{Ti}$. Using then (6.14) and (6.16), we obtain

$$D_i = \lambda \mathbf{I}, \quad \lambda = \zeta \frac{|T|}{n} \tag{6.20}$$

If E_{Ti} is self-conjugate, λ is the efficiency for any contrast of the real vector $\boldsymbol{\alpha}_{Ti}$. If the conjugate of E_{Ti} is a distinct homogeneous subspace E_{Tj}, the information matrix D_j being conjugate of D_i is also equal to $\lambda \mathbf{I}$. Hence λ is the efficiency for any real contrast of $\boldsymbol{\alpha}_T$ which depends only of $\boldsymbol{\alpha}_{Ti}$ and $\boldsymbol{\alpha}_{Tj} = \overline{\boldsymbol{\alpha}}_{Ti}$.

To simplify the notations in the following developement where E_{Ti} is fixed, we let $E = E_{Ti}$, $\chi = \chi_i$ and denote by P the canonical projector on E. The homogeneous space E is supposed to be irreducible. The corresponding eigenvalue ζ of $X_T^* Q_B X_T$ is calculated by the equality

$$\boldsymbol{\tau}^* X_T^* Q_B X_T \boldsymbol{\tau} = \zeta \boldsymbol{\tau}^* \boldsymbol{\tau} , \tag{6.21}$$

valid for any $\boldsymbol{\tau} \in E$. Here $\boldsymbol{\tau}$ is chosen as the projection $P\mathbf{e}_t$ of a vector of the canonical basis of $\mathbb{C}^T$. Before calculating ζ, let us point out that:

(1)The homogeneous subspaces are orthogonal. Hence P is the operator of orthogonal projection on E and satisfies $P^* = P$ and $PP = P$. P commutes with the G-homomorphism $X_T^* Q_B X_T$ (proposition 6.8 and 6.7).

(2) If $\mathbf{s} \in T$ is in the same orbit as $\mathbf{t}$: $\mathbf{s} = g\mathbf{t}$, we have $R_g \mathbf{e}_t = \mathbf{e}_s$, hence since P and R_g commute (proposition 6.7) we have

$$P\mathbf{e}_s = R_g P \mathbf{e}_t \tag{6.22}$$

(3) To the decomposition of T into distinct orbits $T_1, \ldots, T_m$ of G is associated an orthogonal direct sum decomposition $V_1 \oplus \ldots \oplus V_m$ of $\mathbb{C}^T$ into G-invariant subspaces: V_j is the set of vectors (α_t) such that $\alpha_t = 0$ when $\mathbf{t} \notin T_j$. The V_j will be called the *transitive constituent subspaces* and $V_1 \oplus \ldots \oplus V_m$ the *transitive constituent decomposition* of $\mathbb{C}^T$. The irreducible subspace E is included in one of the transitive constituent, say V_j. Then $P\mathbf{e}_t = 0$ if $\mathbf{t} \notin T_j$. Moreover, $P\mathbf{e}_t$ cannot be equal to 0 if $\mathbf{t} \in T_j$. Otherwise $P\mathbf{e}_s$ would

be 0 for every $\mathbf{s} \in T_j$ [by 6.22], P would be 0 and E would also be $\{0\}$ contrary to the assumption of irreducibility.

(4) The treatments belonging to a given orbit T_j are equireplicated. This is so because if $\mathbf{s}$ and $\mathbf{t}$ belong to the same orbit in T and $g \in G$ is such that $\mathbf{s} = g\mathbf{t}$, the map $\mathbf{u} \mapsto g\mathbf{u}$ is a bijection from $\phi_T^{-1}(\mathbf{t})$ onto $\phi_T^{-1}(\mathbf{s})$. We shall denote by r_j the replication of any treatment $\mathbf{t}$ in T_j: $r_j = |\phi_T^{-1}(\mathbf{t})|$.

Proposition 6.10 *For every $\boldsymbol{\tau}$ in V_j, we have $\boldsymbol{\tau}^* X_T^* X_T \boldsymbol{\tau} = r_j \boldsymbol{\tau}^* \boldsymbol{\tau}$.*

Proof. Let $\boldsymbol{\tau}$ be a vector in V_j. We have $\boldsymbol{\tau}(\mathbf{t}) = 0$ for $\mathbf{t} \notin T_j$ and therefore

$$\begin{aligned} \boldsymbol{\tau}^* X_T^* X_T \boldsymbol{\tau} &= \textstyle\sum_{u \in U} |(X_T \boldsymbol{\tau})(u)|^2 = \sum_{u \in U} |\boldsymbol{\tau}\left(\phi_T(u)\right)|^2 \\ &= \textstyle\sum_{\mathbf{t} \in T_j} |\phi_T^{-1}(\mathbf{t})|\ |\boldsymbol{\tau}(\mathbf{t})|^2 = r_j \boldsymbol{\tau}^* \boldsymbol{\tau}\ . \qquad \blacksquare \end{aligned}$$

(5) Let $B_1, \ldots, B_e$ be the distinct orbits of G in B. Partition X_B as $(X_{B1}, \ldots, X_{Be})$ where X_{Bl} contains the columns of X_B indexed by an element $\mathbf{b} \in B_l$. All the blocks in an orbit B_l have the same size k_l, so that $X_{Bl}^* X_{Bl} = k_l \mathbf{I}$, and the operator of orthogonal projection on $\operatorname{Im} X_B$ can be decomposed as:

$$P_B = X_B (X_B^* X_B)^{-1} X_B^* = \sum_l \frac{X_{Bl} X_{Bl}^*}{k_l} \tag{6.23}$$

Calculation of the principal efficiency in E. We choose $\boldsymbol{\tau} = P\mathbf{e_t}$ where $\mathbf{t} \in T_j$ and $E \subset V_j$, so that $\boldsymbol{\tau}$ is not 0. Then

(1) We have

$$\begin{aligned} \boldsymbol{\tau}^* X_T^* X_T \boldsymbol{\tau} &= r_j \boldsymbol{\tau}^* \boldsymbol{\tau} = r_j \mathbf{e}_t^* P \mathbf{e}_t = \tfrac{r_j \chi(1)}{|G|} \textstyle\sum_{g \in G} \overline{\chi}(g) \mathbf{e}_t^* R_g \mathbf{e}_t \\ &= \tfrac{r_j \chi(1)}{|G|} \textstyle\sum_{g \in G} \overline{\chi}(g) \mathbf{e}_t^* \mathbf{e}_{gt} = \tfrac{r_j \chi(1)}{|G|} \sum_{g \in G_\mathbf{t}} \overline{\chi}(g)\ , \end{aligned}$$

where $G_\mathbf{t}$ is the stabilizer of $\mathbf{t}$ in G. The last sum can be further simplified using Frobenius reciprocity theorem ([14], th3.1). Denote by χ_G the restriction of χ to $G_\mathbf{t}$, by $\mathbf{1}_{G_\mathbf{t}}$ the trivial character of $G_\mathbf{t}$ and by $\mathbf{1}_{G_\mathbf{t}}^G$ the character induced by $\mathbf{1}_{G_\mathbf{t}}$ on G. We have

$$\frac{1}{|G_\mathbf{t}|} \sum_{g \in G_\mathbf{t}} \overline{\chi}(g) = \langle \mathbf{1}_{G_\mathbf{t}}, \chi_{G_\mathbf{t}} \rangle_{G_\mathbf{t}} = < \mathbf{1}_{G_\mathbf{t}}^G, \chi >_G \ .$$

Moreover, it follows from the definition that $\mathbf{1}_{G_\mathbf{t}}^G$ is the permutation character π of the operation of G on G/G_t, hence also the permutation character of the operation of G on the orbit T_j of $\mathbf{t}$, which is the character of the G-submodule V_j too. E being a homogeneous irreducible subspace of V_j, the multiplicity of χ in π is 1 so that

$$\left\langle \mathbf{1}_{G_\mathbf{t}}^G, \chi \right\rangle_G = \langle \pi, \chi \rangle_G = 1\ .$$

Finally, we get

$$\boldsymbol{\tau}^* \boldsymbol{\tau} = \frac{\boldsymbol{\tau}^* X_T^* X_T \boldsymbol{\tau}}{r_j} = \frac{\chi(1) |G_\mathbf{t}|}{|G|} = \frac{\chi(1)}{|T_j|} \tag{6.24}$$

(2) $\boldsymbol{\tau}^* X_T^* Q_B X_T \boldsymbol{\tau} = \boldsymbol{\tau}^* X_T^* X_T \boldsymbol{\tau} - \boldsymbol{\tau}^* X_T^* P_B X_T \boldsymbol{\tau}$, where $P_B = I - Q_B$ is the operator of orthogonal projection onto $\operatorname{Im} X_B$, and

$$\begin{aligned}
\boldsymbol{\tau}^* X_T^* P_B X_T \boldsymbol{\tau} &= \mathbf{e}_\mathbf{t}^* P^* X_T^* P_B X_T P \mathbf{e}_\mathbf{t} = \mathbf{e}_\mathbf{t}^* X_T^* P_B X_T P \mathbf{e}_\mathbf{t} \\
&= \tfrac{\chi(1)}{|G|} \textstyle\sum_g \overline{\chi}(g) \mathbf{e}_\mathbf{t}^* X_T^* P_B X_T R_g \mathbf{e}_\mathbf{t} = \tfrac{\chi(1)}{|G|} \textstyle\sum_g \overline{\chi}(g) \mathbf{e}_\mathbf{t}^* X_T^* P_B X_T \mathbf{e}_{g\mathbf{t}} \\
&= \tfrac{\chi(1)}{|G|} \textstyle\sum_g \overline{\chi}(g) q_g \ ,
\end{aligned}$$

with

$$q_g = \mathbf{e}_\mathbf{t}^* X_T^* P_B X_T \mathbf{e}_{g\mathbf{t}} \tag{6.25}$$

The eigenvalue ζ is then

$$\begin{aligned}
\zeta &= \tfrac{\boldsymbol{\tau}^* X_T^* Q_B X_T \boldsymbol{\tau}}{\boldsymbol{\tau}^* \boldsymbol{\tau}} = \tfrac{r_j \boldsymbol{\tau}^* \boldsymbol{\tau} - [\chi(1)/|G|] \sum_g \overline{\chi}(g) q_g}{\boldsymbol{\tau}^* \boldsymbol{\tau}} \\
&= r_j - \tfrac{\sum_g \overline{\chi}(g) q_g}{|G_t|} \ ,
\end{aligned}$$

and the efficiency of any real contrast of $\boldsymbol{\alpha}_{Ti}$ and $\overline{\boldsymbol{\alpha}}_{Ti}$ is given by

$$\lambda = \frac{r_j |T|}{n} - \frac{|T|}{n} \frac{\sum_g \overline{\chi}(g) q_g}{|G_t|} \ . \tag{6.26}$$

If we use formula (6.23) and put $c_{gl} = \mathbf{e}_\mathbf{t}^* X_T^* X_{Bl} X_{Bl}^* X_T \mathbf{e}_{g\mathbf{t}}$, we have $q_g = \sum_l c_{gl}/k_l$; hence

$$\lambda = \frac{r_j |T|}{n} - \frac{|T|}{n} \frac{\sum_l (1/k_l) \sum_g \overline{\chi}(g) c_{gl}}{|G_\mathbf{t}|} \ . \tag{6.27}$$

c_{gl} is the *concurrence* of treatments $\mathbf{t}$ and $g\mathbf{t}$ within the set of blocks B_l, which is homogeneous by proposition 5.5. The concurrence $\mathbf{e}_\mathbf{t}^* X_T^* X_{Bl} X_{Bl}^* X_T \mathbf{e}_\mathbf{s}$ between two treatments in the same orbit can be obtained by examination of an initial block K in B_l. We note first that if B_l is the disjoint union of d cyclic sets with initial block K, the total concurrence is d times the concurrence within one of these cyclic sets. Theorem 4.4 of BR [4] gives then a method to calculate the concurrence of $\mathbf{t}$ and $\mathbf{s}$ within this cyclic set. Let $G(\mathbf{s}, \mathbf{t})$ be the orbit of $(\mathbf{s}, \mathbf{t})$ in G. This concurrence is $\sum K(x) K(y) |G_{\mathbf{st}}| / |G_K|$, where the sum is other pairs (x, y) in $G(\mathbf{s}, \mathbf{t})$, $G_{\mathbf{st}}$ is the stabilizer of $(\mathbf{s}, \mathbf{t})$ in G and G_K the stabilizer of K. Since there is $|G|/|G_K|$ blocks in each cyclic set, $|B_l| = d|G|/|G_K|$ and

$$c_{gl} = \frac{|B_l|}{|G|} \sum_{(x,y) \in G(\mathbf{s},\mathbf{t})} K(x) K(y) |G_{\mathbf{st}}| \ . \tag{6.28}$$

Remarks.

(1) Let R be a system of representatives of the conjugacy classes in G, and for each $r \in R$ let $C(r)$ be the class of r: $C(r) = \{g | g = h^{-1} r h\}$. Since $\chi(g)$ is constant on each conjugacy class, we have

$$\sum_g \overline{\chi}(g) q_g = \sum_{r \in R} \overline{\chi}(r) \sum_{g \in C(r)} q_g \ .$$

Though $q_g = \mathbf{e}_{\mathbf{t}}^* X_T^* P_B X_T \mathbf{e}_{g\mathbf{t}}$ depends on the choice of $\mathbf{t}$ in T_j, the partial sums $\sum_{g \in C(r)} q_g$ do not depend on it. Suppose indeed that $\mathbf{s} = h\mathbf{t}$, where $h \in G$, and put $h(g) = hgh^{-1}$:

$$e_{\mathbf{s}}^* X_T^* P_B X_T \mathbf{e}_{h(g)\mathbf{s}} = e_{\mathbf{s}}^* X_T^* P_B X_T R_h \mathbf{e}_{g\mathbf{t}} = e_{\mathbf{s}}^* R_h X_T^* P_B X_T \mathbf{e}_{g\mathbf{t}} = e_{\mathbf{t}}^* X_T^* P_B X_T \mathbf{e}_{g\mathbf{t}} \ .$$

The third equality uses the identity $R_h^* \mathbf{e}_{\mathbf{s}} = \mathbf{e}_{\mathbf{t}}$ equivalent to $\mathbf{e}_{\mathbf{s}} = R_h \mathbf{e}_{\mathbf{t}}$. Now since $h(g)$ enumerates the class $C(r)$ when g does, we have

$$\sum_{g \in C(r)} e_{\mathbf{t}}^* X_T^* P_B X_T \mathbf{e}_{g\mathbf{t}} = \sum_{g \in C(r)} e_{\mathbf{s}}^* X_T^* P_B X_T \mathbf{e}_{h(g)\mathbf{s}} = \sum_{g \in C(r)} e_{\mathbf{s}}^* X_T^* P_B X_T \mathbf{e}_{g\mathbf{s}} \ .$$

(2) *Case* $r_j = r$, $k_l = k$. Suppose that the treatments are equireplicated and the blocks equal-sized. Let r and k be the common values of the r_j on one side, of the k_l on the other. Then

$$r_j|T| = r|T| = n \ ,$$

$$q_g = \mathbf{e}_{\mathbf{t}}^* X_T^* P_B X_T \mathbf{e}_{g\mathbf{t}} = \frac{1}{k} c_g \ ,$$

where $c_g = \mathbf{e}_{\mathbf{t}}^* X_T^* X_B X_B^* X_T \mathbf{e}_{g\mathbf{t}}$ is the element of coordinate $(\mathbf{t}, g\mathbf{t})$ of the matrix $X_T^* X_B X_B^* X_T$. This element is the concurrence between treatment $\mathbf{t}$ and treatment $g\mathbf{t}$.

The efficiency then become:

$$\lambda = 1 - \frac{1}{rk} \frac{\sum_g \overline{\chi}(g) c_g}{|G_t|} \qquad (6.29)$$

If G is semiregular on T, then $G_{\mathbf{t}} = \{\mathbf{1}\}$ and we find that $\lambda = 1 - \nu$ where $\nu = (1/rk) \sum_g \overline{\chi}(g) c_g$. This is the quantity found for the intrablock efficiency by BR in the case where G is abelian, T is identified with G and the operation is defined by translation: $g\mathbf{t} = g + \mathbf{t}$. Notice that if $|G_t| = \mathbf{1}$, the dimension of E must be 1 and χ is therefore a linear character.

Example. We consider the example 5.3 of BR [4]. The treatments are the ten genotypes of some plant obtained by crossing all pairs of five pure parental lines, but omitting self-crosses and ignoring the gender of the parents. They are identified with the set T of unordered pairs from $\{1, 2, 3, 4, 5\}$. G is the symmetric group S_5 in its action on unordered pairs. We consider the cyclic set Δ generated by the initial block $K = \{\{1, 2\}\{3, 4\}\}$. The elements of the stabilizer of K are the eight permutations generated by $(1, 2)$, $(3, 4)$, $(1, 3)(2, 4)$:

$()$,	$(1, 2)$,	$(3, 4)$,	$(1, 2)(3, 4)$,
$(1, 3)(2, 4)$,	$(1, 4, 2, 3)$,	$(1, 3, 2, 4)$,	$(1, 4)(2, 3)$.

Hence Δ contains $5!/8 = 15$ blocks which are all the pairs of genotypes with no parental lines in common. The concurrence of two treatments is 3 if they are identical, 1 if they have no parental line in common, 0 otherwise. For $i = \{1, 2, 3, 4, 5\}$, define the element v_i in $\mathbb{C}^T$ by $v_i(\mathbf{t}) = 1$ if $i \in \mathbf{t}$, $v_i(\mathbf{t}) = 0$ otherwise. Let E_0 and E be the subspaces of $\mathbb{C}^T$ spanned by $v_1 + v_2 + \cdots + v_5$ and $\{v_1, v_2, v_3, v_4, v_5\}$ respectively. Then it can be shown that the homogeneous decomposition of $\mathbb{C}^T$ is $E_0 \oplus E_1 \oplus E_2$ where $E_1 = E \cap E_0^\perp$ and $E_2 = E^\perp$. These subspaces are irreducible and the associated characters χ_0, χ_1, χ_2 are given in table 2 (using the usual notation for the conjugacy classes which is given in [14]). To calculate the corresponding efficiencies, we can use (6.29), with $r = 3$, $k = 2$. We choose $\mathbf{t} = \{1, 2\}$ (we could choose any other element of T since S_5 is transitive on T). The number of elements of $G_{\mathbf{t}} = G_{\{1,2\}}$ is 12. The number of elements of each conjugacy class such that $c_g = 1$ or $c_g = 3$ is given at the bottom of table 2. The sums $\sum \overline{\chi}(g) c_g$ deduced from these numbers for $\chi = \chi_0, \chi_1, \chi_2$, and the corresponding efficiencies λ_0, λ_1, λ_2, appear on the bottom of table 2.

conjugacy class	No. of elements	χ_0	χ_1	χ_2	No. of $g \in G$ such that $c_g = 1$	$c_g = 3$
1	1	1	4	5	0	1
2	10	1	2	1	0	4
2^2	15	1	0	1	6	3
3	20	1	1	-1	0	2
2.3	20	1	-1	1	12	2
4	30	1	0	-1	6	0
5	24	1	-1	0	12	0
$\sum \overline{\chi}(g) c_g$		72	12	48		
Efficiency		$\lambda_0 = 0$	$\lambda_1 = \frac{5}{6}$	$\lambda_2 = \frac{2}{6}$		

Table 2: Characters of E_0, E_1, E_2 and corresponding efficiencies

To obtain, in each conjugacy class, the numbers of g satisfying $c_g = 1$, note first that the concurrence c_g between $\{1, 2\}$ and $g\{1, 2\} = \{g1, g2\}$ is 1 iff the couple $(g1, g2)$ is one of the following six: $(3, 4)$, $(4, 3)$, $(3, 5)$, $(5, 3)$, $(4, 5)$, $(5, 4)$. Let us select one of these couples, say $(3, 4)$. The corresponding permutations $g \in S_5$, which send 1 to 3 and 2 to 4 are listed in table 3. The similar tables for the other couples are obtained by permutation of the numbers 3, 4, 5. Hence to get the sought numbers, it is sufficient to multiply by 6 the numbers in the last row of table 3. The row for $c_g = 3$ in table 2 is obtained even more simply by noting that $c_g = 3$ if and only if g is one of the 12 permutations of the stabilizer $G_{\{1,2\}}$.

6.3 Efficiency when the irreducible character involved is linear

When the dimension of the irreducible homogeneous space E is 1, that is when χ is a linear character, formula (6.27) can be simplified. The sum $\sum_{(x,y) \in G(\mathbf{s},\mathbf{t})} K(x)K(y)|G_{\mathbf{st}}|$ in (6.28) is readily seen to be the same as

$$\sum_{h \in G} K(h\mathbf{s})K(h\mathbf{t}) \; .$$

Conj.class	Perm. g	number
1		0
2		0
2^2	(13)(24)	1
3		0
2.3	(13)(245) (135)(24)	2
4	(1324)	1
5	(13245) (13524)	2

Table 3: Permutations sending 1 on 3, 2 on 4

Now

$$\sum_g \overline{\chi}(g) c_{gl} = \frac{|Bl|}{|G|} \sum_g \overline{\chi}(g) \sum_h K(h\mathbf{s}) K(h\mathbf{t}) \ .$$

Writing $g' = hg$ and using the fact that $\mathbf{s} = g\mathbf{t}$, this becomes

$$\frac{|B_l|}{|G|} \sum_{g'} \overline{\chi\chi}(h^{-1}g') \sum_h K(g'\mathbf{t}) K(h\mathbf{t}) \ ,$$

which, using the linearity of χ, is equal to

$$\frac{|B_l|}{|G|} \sum_{g'} \sum_h \chi(h) \overline{\chi}(g') K(g'\mathbf{t}) K(h\mathbf{t}) \ ,$$

or

$$\frac{|B_l|}{|G|} \left| \sum_{g'} \overline{\chi}(g') K(g'\mathbf{t}) \right|^2 \ .$$

Substitution in (6.27) now gives

$$\lambda = \frac{r_j |T|}{n} - \frac{|T|}{n} \frac{\sum_l (|B_l|/k_l) |\sum_g \overline{\chi}(g) K_l(g\mathbf{t})|^2}{|G||G_t|} \ . \tag{6.30}$$

The advantage of this formula on (6.27) is that the coefficients of $\overline{\chi}(g)$ in it are directly known and are generally 0 or 1 according to the presence or absence of the treatment $g\mathbf{t}$ in the initial block.

If the stabilizer G_K of a multiset K is not reduced to the identity, the sum $\sum_g \overline{\chi}(g) K(g\mathbf{t})$ can be further simplified. Let indeed $G_K g_1, \ldots, G_K g_b$ be the distinct right cosets of G_K in G. If h belongs to G_K, we have $K(hg_i\mathbf{t}) = K(g_i\mathbf{t})$ (see proof of prop. 5.3). Hence

$$\begin{aligned} \textstyle\sum_g \overline{\chi}(g) K(g\mathbf{t}) &= \textstyle\sum_{h \in G_K} \sum_{i=1}^b \overline{\chi}(hg_i) K(hg_i\mathbf{t}) \\ &= \textstyle\left(\sum_{h \in G_K} \overline{\chi}(h)\right) \left(\sum_i \overline{\chi}(g_i) K(g_i\mathbf{t})\right) \ . \end{aligned} \tag{6.31}$$

The first sum in the last product is equal to $|G_K|$ if the restriction of χ to G_K is the trivial character $\mathbf{1}$, and is 0 otherwise. This follows from the following proposition.

Proposition 6.11 *If χ is a linear character of the group G and H a subgroup of G, the sum $\sum_{h\in H}\overline{\chi}(h)$ is equal to $|H|$ if $\chi_H = \mathbf{1}_H$ and to 0 otherwise.*

χ_H is the restriction of χ to H, and $\mathbf{1}_H$ the trivial character $\mathbf{1}$ on H.

Proof. χ_H and $\mathbf{1}_H$ are linear, hence irreducible. The scalar product $\langle \mathbf{1}_H, \chi_H\rangle = \sum_{h\in H}\overline{\chi}(h)/|H|$ is therefore 1 if $\chi_H = \mathbf{1}_H$ and 0 otherwise ■

For each multiset K, we let

$$Q(K) = \left|\sum_g \overline{\chi}(g)K(g\mathbf{t})\right| \tag{6.32}$$

From (6.31) and the proposition 6.11, it follows that:

$$Q(K) = \begin{cases} |G_K||\sum_i \overline{\chi}(g_i)K(g_i\mathbf{t})| & \text{if } \chi(g)=1 \text{ on } G_K, \\ 0 & \text{otherwise} \end{cases} \tag{6.33}$$

where the g_i are representatives of the distinct right cosets of the stabilizer G_K of K.

The next proposition sums up the preceding results. Let us recall that χ is here a linear character and E the associated homogeneous subspace, which is supposed to be of dimension 1 (i.e. irreducible). E is included in the transitive constituent subspace V_j, associated to an orbit T_j of G in T and $\mathbf{t}$ is an element of T_j.

Proposition 6.12 *The efficiency factor for any contrast $\langle \mathbf{c}, \boldsymbol{\tau}\rangle$ with $c\in E+\overline{E}$ is*

$$\lambda = \frac{r_j|T|}{n} - \frac{|T|}{n}\frac{\sum_l Q(K_l)^2|B_l|/k_l}{|G||G_{\mathbf{t}}|}$$

If G' is the commutator group of G and ψ the canonical surjection from G onto G/G', the linear character χ can be deduced from an irreducible character χ_0 of the abelian group G/G' by the relation $\chi(g) = \chi_0(\psi(g))$ ([14], th 2.8). It is then clear that the restriction of χ to G_K is the trivial character $\mathbf{1}$ if and only if the restriction of χ_0 to $\psi(G_K)$ is the trivial character $\mathbf{1}$.

To find explicitly the characters χ_0 whose restriction to $\psi(G_K)$ is $\mathbf{1}$, we decompose $\psi(G_K)$, G/G' and the quotient group $(G/G')/\psi(G_K)$ as products of cyclic groups, and denote by δ the canonical injection from $\psi(G_K)$ into G/G' and by ϕ the canonical surjection from G/G' onto $(G/G')/\psi(G_K)$. Then any irreducible character χ_0 of G/G' is of the form $\chi_0 = \boldsymbol{\eta}^{g^\times}=$, where $g^\times$ is an element of the dual group $(G/G')^\times$ (see section 2). Moreover

$$\chi_0(\delta g) = \boldsymbol{\eta}^{g^\times}(\delta g) = \eta^{[g^\times,\delta g]} = \eta^{[\delta^\times g^\times, g]} ;$$

hence the restriction of χ_0 to $\psi(G_K) = \operatorname{Im}\delta$ is $\mathbf{1}$ iff $\delta^\times g^\times = 0$. Since $\operatorname{Ker}\delta^\times = \operatorname{Im}\phi^\times$, this is also equivalent to the existence of an element h such that $g^\times = \phi^\times h^\times$. Thus the restriction of χ to G_K is $\mathbf{1}$ if χ is of the form $\boldsymbol{\eta}^{\phi^\times h^\times}\circ\psi$. Then

$$\chi(g_i) = \boldsymbol{\eta}^{\phi^\times h^\times}\circ\psi(g_i) = \eta^{[\phi^\times h^\times,\psi(g_i)]} = \eta^{[h^\times,\phi\circ\psi(g_i)]} = \eta^{[h^\times,h_i]} ,$$

where $h_i = \phi \circ \psi(g_i)$ and the quantity $Q(K)$ is

$$Q(K) = |G_K| \left| \sum_i \eta^{[h^\times, h_i]} K(g_i \mathbf{t}) \right| .$$

EXAMPLE. Suppose there are 3 factors A, B, C at three levels each and nine blocks of size 6. The set T of treatments is identified with the abelian group C^3, where C is the cyclic group of order 3. G is taken equal to T, which acts on itself by translation. A suitable design is then provided by a cyclic set whose initial block K is made up of 2 cosets $G_K{+}t_1$, $G_K{+}t_2$ of the subgroup G_K generated by (111) in $G = T$. If $h_1 = \phi(t_1)$, $h_2 = \phi(t_2)$ are the images of t_1, t_2 by the quotient mapping $\phi : G \to G/G_K$, and $\eta = \exp(2\pi \mathrm{i}/3)$, the efficiency is

$$\lambda = \begin{cases} 1 - \left| \eta^{[h^\times, h_1]} + \eta^{[h^\times, h_2]} \right|^2 /4 & \text{if } \delta^\times g^\times = 0 \text{ and } g^\times = \phi^\times h^\times, \\ 1 & \text{if } \delta^\times g^\times \neq 0 \end{cases}$$

G_K and G/G_K can be identified with C and C^2 respectively so that the matrices of $\delta^\times$ and $\phi^\times$ are

$$C^\times \xleftarrow[{[1\ \ 1\ \ 1]}]{\ \ \delta^\times\ \ } (C^3)^\times \xleftarrow[\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}]{\ \ \phi^\times\ \ } (C^2)^\times$$

Table 4 gives all the scalar products $[h^\times, h]$ for $h^\times \in (C^2)^\times$ and $h \in C^2$, and also the images $g^\times = \phi^\times h^\times$. If we take $h_1 = (00)$, $h_2 = (12)$, the efficiencies are

	$[h^\times, h]$										
$h^\times$ $\quad h:$	00	01	02	10	11	12	20	21	22	$g^\times = \phi^\times h^\times$	
00	0	0	0	0	0	0	0	0	0	000	1
01	0	1	2	0	1	2	0	1	2	011	$B\ C$
02	0	2	1	0	2	1	0	2	1	022	B^2C^2
10	0	0	0	1	1	1	2	2	2	101	$A\quad C$
11	0	1	2	1	2	0	2	0	1	112	$A\ B\ C^2$
12	0	2	1	1	0	2	2	1	0	120	$A\ B^2$
20	0	0	0	2	2	2	1	1	1	202	$A^2\quad C^2$
21	0	1	2	2	0	1	1	2	0	210	A^2B
22	0	2	1	2	1	0	1	0	2	221	A^2B^2C

Table 4: Scalar products $[h^\times, h]$ and images $g^\times = \phi^\times h^\times$

0 for ABC^2 and A^2B^2C
$\frac{3}{4}$ for BC, B^2C^2, AC, AB^2, A^2C^2, A^2B
1 for the other contrasts.

Whatever the choice of G_K, h_1, h_2, it is easy to see that there are at least two non null elements $h_1^\times$ and $h_2^\times$ in $(C^2)^\times$ with associated null efficiency. It is thus impossible to find a cyclic set of the good size allowing the estimation of all factorial effects.

6.4 A case with non irreducible homogeneous subspaces of contrasts

Let $T = \sqcup_{j\in J} T_j$ be the decomposition of T as a disjoint union of orbits for G and $\mathbb{C}^T = \oplus_{j\in J} V_j$ the corresponding (orthogonal) transitive constituent decomposition. If E_{ij} is the homogeneous subspace of V_j of character χ_i, the orthogonal decomposition $\oplus_{ij} E_{ij}$ of $\mathbb{C}^T$ is such that $V_j = \oplus_i E_{ij}$ and $E_{Ti} = \oplus_j E_{ij}$. If E_{ij} is different from 0 for two or more indices j, E_{Ti} is not irreducible and the preceding developement does not apply. However, suppose that only one of the images $X_T(E_{ij})$ for $j \in J$ is nonorthogonal to $\operatorname{Im} X_{Bi}$. Then the formulas of section 6.2 can be applied without modification to each irreducible subspace E_{ij}.

Indeed, X_T sends each V_j into the subspace of functions in $\mathbb{C}^U$ which are null outside $\phi_T^{-1}(T_j)$. Hence the images $X_T(V_j)$ of the different transitive constituents are orthogonal, and the $X_T(E_{ij})$ are also orthogonal (by corollary of Schur's lemma). With the above condition, the projections $Q_B X_T(E_{ij})$ are equally orthogonal. Then the E_{ij} are invariant under $X_T^* Q_B X_T$ and therefore included in an eigenspace of this last operator when they are irreducible. We can then use (6.21) to calculate the corresponding efficiency with $\boldsymbol{\tau} = P\mathbf{e}_\mathbf{t}$, where P is the operator of orthogonal projection on E_{Ti}, $\mathbf{t}$ an element of T_j and therefore $\boldsymbol{\tau}$ a non null vector of $P(V_j) = E_{ij}$. Note that if $X_T(E_{ij})$ is orthogonal to the blocks, we find $r_j|T|/n$ as efficiency (and 1 if all the r_j are equal to r).

It must be noticed that the transitive constituent decomposition of $\mathbb{C}^T$ does generally not correspond to a sensible decomposition of the space of treatment contrasts so that, in the preceding situation, some more calculations are needed to find the efficiency for contrasts of interest. Moreover if some of these last contrasts are assumed to be null, the orthogonality, for a fixed i, between the subspaces $Q_B X_T(E_{ij})$ can be destroyed and the formulas of section 6.2 therefore made invalid.

In section 7, a situation of this kind will be studied. T will be a commutative group and G a subgroup of T operating by translation on T.

6.5 Upper bound for the efficiency

Let $E_{Ti})_{i\in I}$ be a family, stable for the conjugation, of homogeneous subspaces of $\mathbb{C}^T$ and $E_I = \oplus_{i\in I} E_{Ti}$ their sum, the dimension of which is denoted by q. Using proposition 4.2, we are going to give an upper bound for the efficiency $\operatorname{eff}(E_{Ir})$ where E_{Ir} is the space of real contrasts $c^*\boldsymbol{\tau}$ with $c \in E_I \cap \mathbb{R}^T$.

We denote by $X_I = (X_T A_{Ti})_{i\in I}$ the part of the design matrix X in model (6.11) associated to E_I, and by $\boldsymbol{\alpha}_I = (\boldsymbol{\alpha}_{Ti})_{i\in I}$ the corresponding subvector of treatment parameters. It follows from (6.9) that E_{Ir} is the space of real contrasts of $\boldsymbol{\alpha}_I$. As in section 3.3, define a real vector $\boldsymbol{\beta}_I = N\boldsymbol{\alpha}_I$, by means of an invertible matrix N having conjugated columns associated to conjugate parameters. E_{Ir} is then the set of all contrasts $\langle \mathbf{a}, \boldsymbol{\beta}_I \rangle$ with $\mathbf{a} \in \mathbb{R}^q$.

Suppose that A_T satisfies (6.16) and that N is unitary. The efficiency can then be defined as $\mathrm{eff}(E_{Ir}) = \psi(\mathbf{I}_q)/\psi(C)$ where C, the per unit information matrix for $\boldsymbol{\beta}_I$, is defined as in (4.3) with X_I instead of X_1 and ψ is a function satisfying the condition (a), (b), (c) of section 4.1.

The blocks $X_T A_{Ti}$, which belong to distinct homogeneous subspaces of $\mathbb{C}^U$, are orthogonal to each other. Consequently $X_I^* X_I$ is block diagonal, with the blocks $A_{Ti}^* X_T^* X_T A_{Ti}$ on the diagonal, and

$$\begin{aligned}\mathrm{trace}(X_I^* X_I) &= \textstyle\sum_{i\in I} \mathrm{trace}(A_{Ti}^* X_T^* X_T A_{Ti}) = \sum_{i\in I} \mathrm{trace}(X_T A_{Ti} A_{Ti}^* X_T^*) \\ &= \textstyle\sum_{i\in I} \mathrm{trace}(|T| X_T P_i X_T^*)\end{aligned}$$

where $P_i = A_{Ti}(A_{Ti}^* A_{Ti})^{-1} A_{Ti}^* = A_{Ti} A_{Ti}^* / |T|$ is the operator of orthogonal projection on the homogeneous subspace $E_{Ti} = \mathrm{Im}\, A_{Ti}$. The column associated to unit $\mathbf{u}$ in X_T^* is the canonical vector $\mathbf{e_t}$ of $\mathbb{C}^T$ having 1 in position $\mathbf{t} = \phi_T(\mathbf{u})$, 0 elsewhere. Hence

$$\mathrm{trace}(|T| X_T P_i X_T^*) = |T| \sum_{\mathbf{t}\in T} \left|\phi_T^{-1}(\mathbf{t})\right| \mathbf{e}_\mathbf{t}^* P_i \mathbf{e_t} \ .$$

It follows from (6.22) that $\mathbf{e}_\mathbf{t}^* P_i \mathbf{e_t}$ is constant in every orbit T_j of G in T. We denote by p_{ij} the corresponding value, which we are now going to calculate.

As in section 6.3, let $T = \sqcup_{j\in J} T_j$ be the decomposition of T as a disjoint union of orbits, $C^T = \oplus_{j\in J} V_j$ the corresponding (orthogonal) transitive constituent decomposition and E_{ij} the homogeneous subspace of V_j of character χ_i. Let P_{ij} be the operator of orthogonal projection on E_{ij}. For $\mathbf{t} \in T_j$, $\mathbf{e_t} \in V_j$. Therefore $\mathbf{e}_\mathbf{t}^* P_{ij} \mathbf{e_t} = \mathbf{e}_\mathbf{t}^* P_i \mathbf{e_t} = p_{ij}$. For $\mathbf{t} \notin T_j$, $e_\mathbf{t}^* P_{ij} \mathbf{e_t} = 0$. Hence

$$\dim(E_{ij}) = \mathrm{trace}(P_{ij}) = \sum_{\mathbf{t}\in T} \mathbf{e}_\mathbf{t}^* P_{ij} \mathbf{e_t} = |T_j| p_{ij} \ .$$

This gives $p_{ij} = \dim(E_{ij})/|T_j|$ and

$$\mathrm{trace}(|T| X_T P_i X_T^*) = |T| \sum_{j_i n J} \frac{\left|\phi_T^{-1}(T_j)\right| \dim(E_{ij})}{|T_j|}$$

. Finally,

$$\begin{aligned}\mathrm{trace}(X_I^* X_I) &= \textstyle\sum_{i\in I} \frac{|T| \sum_{j\in J} |\phi_T - 1(T_j)| \dim(E_{ij})}{|T_j|} \\ &= \textstyle\sum_{j\in J} \frac{|T||\phi_T^{-1}(T_j)|(\sum_{i\in I} \dim(E_{ij}))}{|T_j|} \ .\end{aligned}$$

Put $n_j = |\phi_T^{-1}(T_j)|$, $q_j = \sum_{i\in I} \dim(E_{ij})$. n_j is the number of units receiving a treatment in T_j and q_j is the dimension of $E_I \cap V_j$ (as a complex vector space). By proposition 3.2, it is also the dimension of $E_I \cap V_j \cap \mathbb{R}^T$, that is the dimension of the subspace of contrasts in E_{Ir} which depends only of the effects of treatments in T_j. With the above notations, we thus have:

Proposition 6.13 trace$(X_I^* X_I) = \sum_{j\in J} n_j |T| q j_/ |T_j| \leq nq\beta$ *where* $\beta = \max_{j\in J} |T| q_j / |T_j| q$. *We have* $\beta \geq 1$, *with equality if and only if the ratios* $q_j/|T_j|$ *are all equal. In particular* $\beta = 1$ *if* G *operates transitively on* T.

Proof. Indeed, $\sum q_j = \sum_{i\in I} \sum_j \dim(E_{ij}) = \sum_{i\in I} \dim(E_{Ti}) = q$ and $\sum |T_j| = |T|$. If the ratios $q_j/|T_j|$ are all equal, they are also equal to $q/|T| = (\sum_j q_j)/(\sum_j |T_j|)$ and $\beta = 1$. Conversely, if $\beta = 1$ the ratios $q_j/|T_j|$ are equal to $q/|T|$. Moreover, if β were stricly less then 1, we would have $|T| q_j < |T_j| q$ for all j; hence by summing over j, $|T| q < |T| q$ which is absurd ■

Proposition 4.2 implies that $\psi(C) \geq \psi(\beta \mathbf{I}_q)$, hence

$$\text{eff}(E_{Ir}) = \frac{\psi(\mathbf{I}_q)}{\psi(C)} \leq \frac{\psi(\mathbf{I}_q)}{\psi(\beta \mathbf{I}_q)} .$$

The property (b) of section 4.1 shows that the last ratio is greater or equal to 1. It is of course equal to 1 if $\beta = 1$.

7 Balancing the loss of information in factorial block designs

From now on T is a commutative (additive) group and G a subgroup operating on T by translation. The G-set of units is $U = G_t imes V$ with the operation $h(g, v) = (h + g, v)$. The G-morphism ϕ_T is defined by

$$\phi_T(g, v) = t_v + \phi(g) \tag{7.34}$$

where ϕ is the canonical injection from G into T. The elements t_v are chosen so that the family $(t_v + G)_{v\in V}$ includes r times each of the cosets of G in some subgroup T_1 containing G. The design thus includes r replicates of the fraction T_1 of T. In most applications, r will be 1.

The G-set of blocks is a disjoint union $B = \sqcup_{v\in V} B_v$ of quotient groups B_v of G. We write ϕ_{Bv} for the canonical surjection of G onto B_v. Then ϕ_B is defined by

$$\phi_B(g, v) = \phi_{Bv}(g) . \tag{7.35}$$

The set $G \times \{v\}$ of units belonging to one of the blocks in B_v is called a *macroblock*.

The commutative groups involved are decomposed as products of cyclic groups. The same notation is then used for group morphisms, or elements of a group, and their representations in these decompositions (but we write ϕg instead of $\phi(g)$ when representations are concerned).

The linear model of the cyclic design (ϕ_T, ϕ_B) is

$$E(y(g, v)) = \boldsymbol{\tau}(t_v + \phi(g)) + \xi_v(\phi_{Bv}(g)) \tag{7.36}$$

The vector $\boldsymbol{\tau}$ of treatment effects is supposed to satisfy (2.7), with parameters $\alpha_{\mathbf{t}^\times}$, $\mathbf{t}^\times \in S^\times$, which are unconfounded on the fraction T_1. In other words, if γ is the canonical injection from T_1 into T, then $\gamma^\times$ is supposed to be injective on $S^\times$.

$\boldsymbol{\xi}_v$ is the vector of block effects in B_v. It is decomposed on the orthogonal basis of irreducible characters of the group B_v:

$$\boldsymbol{\xi}_v = \sum_{\mathbf{b} \in B_v^\times} \alpha_{\mathbf{b}^\times} \boldsymbol{\eta}^{\mathbf{b}^\times} \tag{7.37}$$

(7.36) is equivalent to:

$$E(y(g,v)) = \sum_{\mathbf{t}^\times \in S^\times} \alpha_{\mathbf{t}^\times} \eta^{[\mathbf{t}^\times, \mathbf{t}_v + \phi g]} + \sum_{\mathbf{b}^\times \in B_v^\times} \alpha_{\mathbf{b}^\times} \eta^{[\mathbf{b}^\times, \phi_{Bv} g]} ,$$

hence to

$$E(y(g,v)) = \sum_{\mathbf{t}^\times \in S^\times} \alpha_{\mathbf{t}^\times} \eta^{[\mathbf{t}^\times, \mathbf{t}_v][\phi^\times \mathbf{t}^\times, g]} + \sum_{\mathbf{b} \in B_v^\times} \alpha_{\mathbf{b}^\times} \eta^{[\phi_{Bv}^\times \mathbf{b}^\times, g]} \tag{7.38}$$

Denote by $B^\times$ the disjoint union of the $B_v^\times$: $B^\times = \sqcup_{v \in V} B_v^\times$. Let Z_T and Z_B be the matrices, of dimensions $|U| \times |S^\times|$ and $|U| \times |B^\times|$ respectively, defined by

$$Z_T\left((g,v), \mathbf{t}^\times\right) = \eta^{[\mathbf{t}^\times, \mathbf{t}_v]} \eta^{[\phi^\times \mathbf{t}^\times, g]} , \tag{7.39}$$

$$Z_B\left((g,v), \mathbf{b}^\times\right) = \begin{cases} \eta^{[\phi_{Bv}^\times \mathbf{b}^\times, g]} & \text{if } \mathbf{b}^\times \in B_v^\times, \\ 0 & \text{otherwise} \end{cases} \tag{7.40}$$

In matrix form, (7.38) becomes

$$E(y) = Z_T \boldsymbol{\alpha}_T + Z_B \boldsymbol{\alpha}_B . \tag{7.41}$$

Denote by $Z_T(, \mathbf{t}^\times)$ the column $\mathbf{t}^\times$ of Z_T and similarly by $Z_B(, \mathbf{b}^\times)$ the column $\mathbf{b}^\times$ of Z_B. To study the model (7.41), we must determine the elements of $Z_T^* Z_T$, $Z_T^* Z_B$, $Z_B^* Z_B$, which are the scalar products Here

(1) we have

$$[Z_T(., \mathbf{t}_1^\times), Z_T(., \mathbf{t}_2^\times)] = \begin{cases} r|T_1| = n & \text{if } \mathbf{t}_1^\times = \mathbf{t}_2^\times \\ 0 & \text{otherwise} \end{cases} \tag{7.42}$$

This is because the design is formed by r copies of the fractional set T_1 and $\gamma^\times$ is injective on $S^\times$.

(2)If $\mathbf{b}^\times \in B_v$,

$$[Z_T(., \mathbf{t}^\times), Z_B(., \mathbf{b}^\times)] = \eta^{[\mathbf{t}^\times, \mathbf{t}_v]} \sum_g \eta^{[\phi^\times \mathbf{t}^\times - \phi_{Bv}^\times \mathbf{b}^\times, g]} ,$$

$$[Z_T(., \mathbf{t}^\times), Z_B(., \mathbf{b}^\times)] = \begin{cases} |G| \eta^{[\mathbf{t}^\times, \mathbf{t}_v]} & \text{if } \phi^\times \mathbf{t}^\times - \phi_{Bv}^\times \mathbf{b}^\times = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7.43}$$

(3) If $\mathbf{b}_1^\times$ and $\mathbf{b}_2^\times$ do not belong to the same subset $B_v^\times$ of $B^\times$, then $[Z_B(.,\mathbf{b}_1^\times), Z_B(.,\mathbf{b}_2^\times)]$ is null. If $\mathbf{b}_1^\times$ and $\mathbf{b}_2^\times$ both belong to $B_v^\times$, we have

$$[Z_B(.,\mathbf{b}_1^\times), Z_B(.,\mathbf{b}_2^\times)] = \sum_g \eta^{[\phi_{Bv}^\times(\mathbf{b}_1^\times - b_2^\times), g]} .$$

Since ϕ_{Bv} is surjective and its dual $\phi_{Bv}^\times$ consequently injective, the sum is equal to $|G|$ if $\mathbf{b}_1^\times = \mathbf{b}_2^\times$, to 0 otherwise. Hence

$$[Z_B(.,\mathbf{b}_1^\times), Z_B(.,\mathbf{b}_2^\times)] = \begin{cases} |G| & \text{if } \mathbf{b}_1^\times = \mathbf{b}_2^\times \\ 0 & \text{otherwise} \end{cases} \tag{7.44}$$

(7.41) can be written in a form analogous to (6.12):

$$E(\mathbf{y}) = X\boldsymbol{\alpha} = \sum_{g^\times} X_{g^\times}\boldsymbol{\alpha}_{g^\times} = \sum_{g^\times}(X_{Tg^\times}\boldsymbol{\alpha}_{Tg^\times} + X_{Bg^\times}\boldsymbol{\alpha}_{Bg^\times}) , \tag{7.45}$$

where

- $X_{Tg^\times}$ contains the columns of Z_T associated to the elements $\mathbf{t}^\times$ of $S^\times$ which have $g^\times$ as image by $\phi^\times$,
- $X_{Bg^\times}$ contains the columns of Z_B associated to the elements $\mathbf{b}^\times$ of $B^\times$ which have $g^\times$ as image by $\phi_{Bv}^\times$, where v is such that $\mathbf{b}^\times \in B_v$,
- $X_{g^\times} = (X_{Tg^\times}, X_{Bg^\times})$,
- X is the matrix with the blocks $X_{g^\times}$ put side by side: $X = (X_{g^\times}, g^\times \in G^\times)$.

The matrix X^*X is block diagonal: $X^*X = \mathrm{diag}(X_{g^\times}^* X_{g^\times}, g^\times \in G^\times)$. Let J_1 be a subset of $S^\times$ satisfying (3.16) and E be the subspace of real contrasts of the parameters $\alpha_{\mathbf{t}^\times}$, $\mathbf{t}^\times \in J_1$.tTo find the principal efficiencies and contrasts in E, we can proceed first separately within each block of X. We therefore fix $g^\times$ and put

$$L = \{\mathbf{t}^\times \in S^\times | \phi^\times(\mathbf{t}^\times) = g^\times\} ,$$

$$L_1 = \{\mathbf{t}^\times \in J_1 | \phi^\times(\mathbf{t}^\times) = g^\times\},$$

$$L_0 = \{\mathbf{t}^\times \in S^\times - J_1 | \phi^\times(\mathbf{t}^\times) = g^\times\} = L - L_1 ,$$

$$X_{B0} = X_{Bg^\times} ,$$

$$X_{T0} = (X_{Tg^\times}(,\mathbf{t}^\times), \mathbf{t}^\times \in L_0) ,$$

$$X_0 = (X_{B0}, X_{T0}) ,$$

$$X_1 = (X_{Tg^\times}(, \mathbf{t}^\times), \mathbf{t}^\times \in L_1) .$$

X_1 is thus the submatrix of $X_{Tg^\times}$ associated to parameters in E, while X_0 is the complementary submatrix, decomposed in two parts X_{T0}, X_{B0} corresponding to treatment effects and block effects respectively. (7.42), (7.43) and (7.44) show that, up to a reordering of its columns and (the same reordering on its) rows, the matrix $X^*_{Tg^\times} X_{Tg^\times}$ has the form

$$\begin{bmatrix} X_1^* X_1 & X_1^* X_{T0} & X_1^* X_{B0} \\ X_{T0}^* X_1 & X_{T0}^* X_{T0} & X_{T0}^* X_{B0} \\ X_{B0}^* X_1 & X_{B0}^* X_{T0} & X_{B0}^* X_{B0} \end{bmatrix} = \begin{bmatrix} n\mathbf{I} & 0 & |G|A_1^* \\ 0 & n\mathbf{I} & |G|A_0^* \\ |G|A_1 & |G|A_0 & |G|\mathbf{I} \end{bmatrix}$$

where if $\mathbf{b}^\times \in B_v$ and $\phi_{Bv}(\mathbf{b}^\times) = g^x x$,

$$\begin{aligned} A_1(\mathbf{b}^\times, \mathbf{t}^\times) &= \eta^{[\mathbf{t}^\times, \mathbf{t}_v]}, \qquad \mathbf{t}^\times \in L_1 . \\ A_0(\mathbf{b}^\times, \mathbf{t}^\times) &= \eta^{[t?, tv]}, \qquad \mathbf{t}^\times \in L_0 . \end{aligned}$$

The per unit information matrix is $D = X_1^* Q_0 X_1 / n$ where Q_0 is the operator of orthogonal projection on the orthogonal complement of $\operatorname{Im} X_0$. Since X_1 and X_{T0} are orthogonal, $Q_0 X_1$ can be obtained as follows:

(1) Orthogonalize X_{B0} for X_{T0}, which leads to

$$W = \left(\mathbf{I} - \frac{X_{T0} X_{T0}^*}{n}\right) XB0 .$$

(2) Orthogonalize X_1 for W, which leads to

$$Q_0 X_1 = \left[\mathbf{I} - W(W^*W)^- W^*\right] X_1 .$$

Finally,

$$D = \frac{X_1^*(\mathbf{I} - W(W^*W)^- W^*) X_1}{n} = \frac{X_1^* X_1}{n}) - \frac{X_1^* W (W^*W)^- W^* X_1}{n} .$$

Since $X_{T0}^* X_1 = 0$, we have

$$W^* X_1 = X_{B0}^* X_1 = |G| A_1$$

$$W^* W = X_{B0}^* \left(\mathbf{I} - \frac{X_{T0} X_{T0}^*}{n}\right) X_{B0} = |G|\mathbf{I} - \frac{|G|^2 A_0 A_0^*}{n}$$

$$D = \mathbf{I} - \frac{|G|}{n} A_1^* B_0^- A_1, \quad \text{with } B_0 = \mathbf{I} - \tfrac{|G|}{n} A_0 A_0^* .$$

Suppose now that X_{B0} has only one column. This is equivalent to the fact that *there is only one index v in V such that ${\phi_{Bv}^\times}^{-1}(g^\times$ is not empty* (and is also equivalent to the hypothesis formulated in section 6.4). Then:

(1) $A_0 A_0^* = |L_0|$, $A_1 A_1^* = |L1|$
(2) B_0 is a scalar: $B_0 = 1 - |G||L_0|/n$
(3) A_1^* is an eigenvector of D with corresponding eigenvalue λ:

$$\lambda = 1 - \frac{|G||L_1|/n}{1 - |G||L_0|/n} \tag{7.46}$$

The orthogonal complement of A_1^* in $\mathbb{C}^{|L1|}$ is an eigenspace of D with corresponding eigenvalue 1.

EXAMPLE. Suppose there are seven factors $A, \ldots, G$ at two levels each, $2^6 = 64$ experimental units grouped in blocks of 4. We would like a design of resolution V, i.e. allowing the estimation of all effects in a model containing the main effects and the interactions between two factors.

Let $C = [0,1]$ be the cyclic group of order 2. T is identified with C^7. Then $T^1 = \operatorname{Ker} \psi_1$ and $G = \operatorname{Ker} \psi_G$ where

$$\psi_1 = [1111111]\,,$$

$$\psi_G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}\,.$$

The elements of $T^\times$ are written in the usual multiplicative way: AB for (1100000) for instance. Thus $\operatorname{Im} \psi_G^\times$ contains the following elements:

$$\begin{array}{ll} \mathbf{1} & ABCDEFG \\ A\,CDE & \;B\;\;FG \\ \;BCD\,F & A\;\;E\,G \\ AB\;\;EF & \;\;CD\;\;G \end{array}$$

This image is also the kernel of $\phi^\times$, where $\phi : G \to T$ is the canonical injection. By hypothesis, we have

$$\begin{aligned} S^\times = \;& \{A, B, C, D, E, F, G, AB, AC, AD, AE, AF, AG, \\ & BC, BD, BE, BF, BG, CD, CE, CF, CG, DE, DF, DG, EF, EG, FG\} \end{aligned}$$

The sets $L(g^\times) = \{\mathbf{t}^\times \in S^\times | \phi^\times(\mathbf{t}^\times) = g^\times\}$ are therefore

$$\begin{array}{l} \{1\}\{A, EG\}\{B, FG\}\{C, DG\}\{D, CG\}\{E, AG\}\{F, BG\} \\ \{G, BF, AE, CD\}\{AB, EF\}\{AC, DE\}\{AD, CE\}\{AF, BE\} \\ \{BC, DF\}\{BD, CF\} \end{array}$$

To define the morphisms ϕ_{Bv}, we identify G with C^4. This is done so that ϕ has the following matrix (the columns of which generate $G = \operatorname{Ker} \psi_G$):

$$\phi = \begin{bmatrix} 1000 \\ 0100 \\ 0010 \\ 0001 \\ 1011 \\ 0111 \\ 0011 \end{bmatrix}$$

One can define five different morphisms ϕ_{Bv} on C^4 so that the images of their dual $\phi_{Bv}^{\times}$ have only 0 as intersection, for instance,

$$\phi_{B1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \phi_{B2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\phi_{B3} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\phi_{B4} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \phi_{B5} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

We need only four of them. The efficiency for an individual contrast $\boldsymbol{\alpha}_{\mathbf{t}^\times}$ is 1 if $\phi^\times(\mathbf{t}^\times)$ does not belong to $\sqcup_{v \in V} \operatorname{Im} \phi_{Bv}^{\times}$, that is to say if the contrast is unconfounded with the blocks in any of the macro-blocks $G \times \{v\}$. If it is confounded if one of these macro-blocks, the efficiency λ is given by (7.46), where $|L_1| = 1$ and $|L_0| = |L(g^\times)| - |L_1|$. Since $\lambda = \frac{2}{3}$ if $|L(g^\times)| = 2$ and $\lambda = 0$ if $|L(g^\times)| = 4$, it is advisable to select the four morphisms ϕ_{Bv} so that the contrasts in $\{G, BF, AE, CD\}$ are unconfounded. The image of these contrasts by $\phi^\times$ is $g^\times = (0011)$ which belongs to $\operatorname{Im} \phi_{B2}^{\times}$. Hence we choose $V = \{1, 3, 4, 5\}$. The family $(\mathbf{t}_v)_{v \in V}$ can be any set of representatives of the cosets of G into T_1, for instance:

$$\mathbf{t}_1 = (0010000) \quad \mathbf{t}_3 = (0011100)$$
$$\mathbf{t}_4 = (0011010) \quad \mathbf{t}_5 = (0011001)$$

The efficiencies are then
- 1 for the contrasts in $\{C, DG\}$, $\{D, CG\}$, $\{G, BF, AE, CD\}$
- 2/3 for the other contrasts.

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